UTMC Senior Team Round Solutions

UTMC Committee

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1. David is playing a casino game. Each round, he has a $\frac{1}{3}$ chance of winning a jackpot revenue of 2000 tickets. If he does not win this jackpot, he wins a revenue of either 5 tickets or 15 tickets, with an equal chance of winning each prize. If David plays exactly 3 times, and the number of tickets he owns does not change between games, compute the probability that his total revenue is at least 2020 tickets.

   Solution:

   First, if David wins 2 or 3 jackpots, he earns more than $2 \cdot 2000 = 4000 > 2020$ tickets. He has a $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$ chance of winning 3 jackpots, and he has a $\left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \cdot \binom{3}{2} = \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \cdot 3 = \frac{2}{9}$ chance of winning exactly 2 jackpots. Additionally, if David does not win any jackpots, he earns at most $3 \cdot 15 = 45 < 2020$ tickets.
   Now, suppose that David wins exactly one jackpot. This has a $\frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 \cdot \binom{3}{1} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 \cdot 3 = \frac{4}{9}$ chance of happening. However, if he wins two prizes of 5 tickets, he earns $2000 + 5 + 5 = 2010$ tickets, which is not enough. Thus, he needs to win at least one prize of 15 tickets. By complementary counting, this has a $1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$ chance of happening, since he has a $\frac{1}{2}$ chance of winning a prize of 5 tickets whenever he does not win a jackpot. Overall, the probability that David wins exactly one jackpot and earns at least 2020 tickets is $\frac{4}{9} \cdot \frac{3}{4} = \frac{1}{3}$.

   $\therefore$ The total probability that David wins at least 2020 tickets is $\frac{1}{27} + \frac{2}{9} + \frac{1}{3} = \boxed{\frac{16}{27}}$.

2. We have 3 distinct prime numbers p, q, and r such that $p+q+r=50$ and $pq+qr+rp$ is the maximum possible for all such triplets of primes. Find pqr.

   Solution:

   Note that since $p+q+r=50$, at least one of p, q, r must be even. Since there exists only one even prime, we will WLOG set $p=2$. Now, we have that $q+r=50$, and we wish to maximize $qr+2q+2r=qr+100$. Note that by AM-GM, this sum is maximized when $|q-r|$ is as small as possible, which occurs with the pair of primes $(19, 29)$. Thus, we have that our answer is:

$$\begin{aligned} pqr &= 2 \cdot 19 \cdot 29 \\ &= \boxed{1102} \end{aligned}$$

3. A tower with length and width of 2 and height of 100 is constructed with 1 by 1 by 1 blocks. Then, a colour of either red or blue assigned at random, with each block being colored 1 color. What is the expected number of adjacent pairs of blocks with opposite colours?

   Solution:

   We use linearity of expectation. For any adjacent pair of blocks, the probability that they are of different colors is $\frac{1}{2}$. Now, the number of adjacent pairs per "level" is 4, and the number of "vertical" adjacent pairs is "$4 \cdot 99$. This gives us that the number of adjacent pairs in the tower is:

$$\begin{aligned} 4 \cdot 99 + 100 \cdot 4 &= 4 \cdot 199 \\ &= 796 \end{aligned}$$

   Thus, by linearity of expectation we have that the expected number of adjacent pairs of blocks with opposite colors is:

$$796 \cdot \frac{1}{2} = \boxed{398}$$

4. The sequence (x_n) is defined by $x_0 = 1$ and $x_{n+1} = 2022^n + x_n$ for all $n \geq 0$. Compute the last two digits of x_{2020}.

Solution:

Define $a_n = \dfrac{x_n}{2022^n}$. Then, we have $2022a_{n+1} = a_n + 1$ for all $n \geq 0$.

This rearranges to $2022\left(a_{n+1} - \dfrac{1}{2021}\right) = a_n - \dfrac{1}{2021}$, so we define $b_n = a_n - \dfrac{1}{2021}$.

Therefore, $2022b_{n+1} = b_n$, so $b_n = b_0 \cdot 2022^{-n}$ and $a_n = b_0 \cdot 2022^{-n} + \dfrac{1}{2021}$

$x_n = 2022^n\left(b_0 \cdot 2022^{-n} + \dfrac{1}{2021}\right) = b_0 + \dfrac{2022^n}{2021}$

Since $x_0 = b_0 + \dfrac{1}{2021} = 1$, we have $b_0 = \dfrac{2020}{2021}$ and $x_n = \dfrac{2022^n + 2020}{2021}$.

Now, we wish to find the modular inverse of 2021 mod 100. We note that this is equivalent to finding the modular inverse of $3 \cdot 7 = 21$ mod 100

Since $3 \cdot 67 = 201 \equiv 1$ mod 100 and $7 \cdot 43 = 301 \equiv 1$ mod 100, we see that 21^{-1} mod $100 = 67 \cdot 43 = 2881 \equiv 81$ mod 100.

$x_{2020} \equiv 81\left(2022^{2020} + 2020\right) \equiv 81\left(22^{2020} + 20\right)$ mod 100.

Note that $22^{2020} \equiv 2^{2020} \cdot 11^{2020} \equiv 2^{2020} \cdot 11^{2020 \bmod \phi(100)} \equiv 2^{2020} \cdot 11^{20}$ mod 100.

Since $2^{2020} \equiv 0$ mod 4 and $2^{2020} \equiv 2^{2020 \bmod \phi(25)} \equiv 1$ mod 25, $2^{2020} \equiv 76$ mod 100.

Since $20 \equiv 0$ mod 10, $11^{20} = (10+1)^{20} = \displaystyle\sum_{i=0}^{20}\left(\binom{20}{i}10^i\right) \equiv 1$ mod 100.

Therefore, $22^{2020} = 2^{2020} \cdot 11^{2020} \equiv 76$ mod 100.

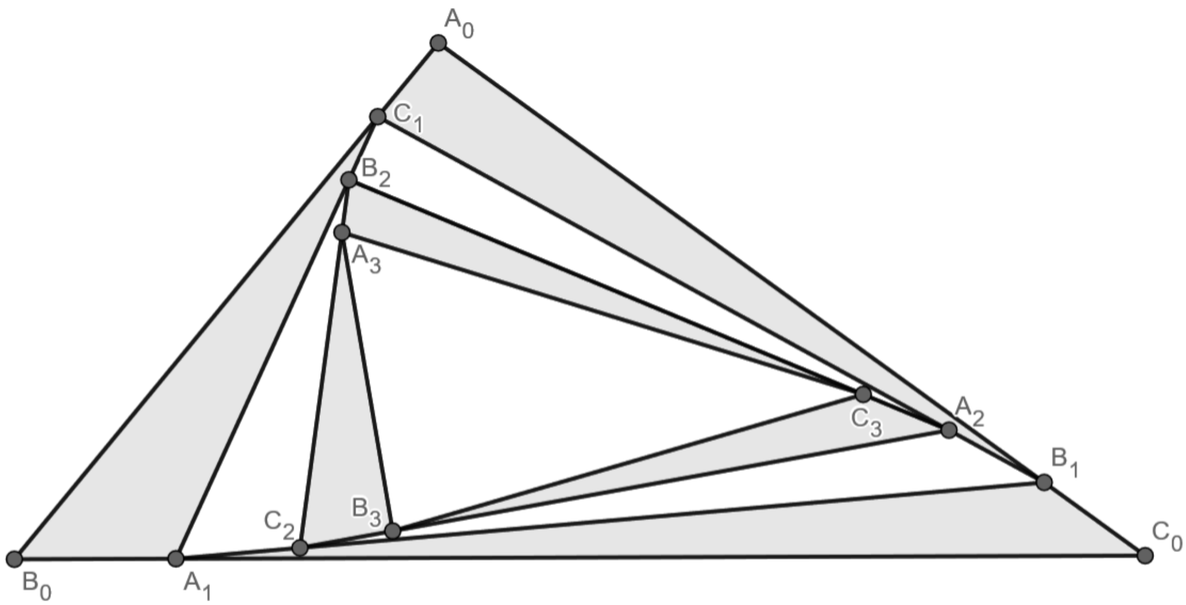
$x_{2020} \equiv 81(76 + 20) \equiv \boxed{76}$ mod 100

5. Let triangle $\Delta A_0B_0C_0$ have an area of 400. For all integers $k > 0$, construct $\Delta A_kB_kC_k$ recursively as follows: let A_k, B_k, and C_k be points on sides $B_{k-1}C_{k-1}$, $C_{k-1}A_{k-1}$, and $A_{k-1}B_{k-1}$, respectively, such that:

$$\frac{A_kB_{k-1}}{A_kC_{k-1}} = \frac{B_kC_{k-1}}{B_kA_{k-1}} = \frac{C_kA_{k-1}}{C_kB_{k-1}} = \frac{1}{6}$$

Compute the total area covered by all the points that are inside an odd number of these triangles.

Solution:



As shown in the diagram, we can find the shaded area by subtracting the area of $\Delta A_1B_1C_1$ from the area of $\Delta A_0B_0C_0$, then adding the area of $\Delta A_2B_2C_2$, then subtracting the area of $\Delta A_3B_3C_3$, and so on. Thus, if we know the ratio between the areas of two consecutive triangles, then we can express the shaded area as an infinite geometric series with a negative common ratio.

For any positive integer k, we are given that:

$$\frac{A_kB_{k-1}}{A_kC_{k-1}} = \frac{1}{6}$$

We can use this to get:

$$\frac{B_{k-1}C_{k-1}}{A_kC_{k-1}} = \frac{A_kC_{k-1} + A_kB_{k-1}}{A_kC_{k-1}} = 1 + \frac{1}{6} = \frac{7}{6}$$

Thus, $A_kC_{k-1} = \frac{6}{7}B_{k-1}C_{k-1}$, and $A_kB_{k-1} = B_{k-1}C_{k-1} - A_kC_{k-1} = \frac{1}{7}B_{k-1}C_{k-1}$. Similarly, $B_kA_{k-1} = \frac{6}{7}C_{k-1}A_{k-1}$, $B_kC_{k-1} = \frac{1}{7}C_{k-1}A_{k-1}$, $C_kB_{k-1} = \frac{6}{7}A_{k-1}B_{k-1}$, and $C_kA_{k-1} = \frac{1}{7}A_{k-1}B_{k-1}$.

Now, let $|\Delta XYZ|$ denote the area of an arbitrary triangle XYZ. Then, we can use the well-known formula $|\Delta XYZ| = \frac{1}{2}XY \cdot XZ \cdot \sin\angle YXZ$ to get:

$$\frac{|\Delta A_{k-1}B_kC_k|}{|\Delta A_{k-1}B_{k-1}C_{k-1}|} = \frac{\frac{1}{2}A_{k-1}B_k \cdot A_{k-1}C_k \cdot \sin\angle B_{k-1}A_{k-1}C_{k-1}}{\frac{1}{2}A_{k-1}C_{k-1} \cdot A_{k-1}B_{k-1} \cdot \sin\angle B_{k-1}A_{k-1}C_{k-1}} = \frac{6}{7} \cdot \frac{1}{7} = \frac{6}{49}$$

Thus, $|\Delta A_{k-1}B_kC_k| = \frac{6}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|$.

Similarly, $|\Delta B_{k-1}C_kA_k| = |\Delta C_{k-1}A_kB_k| = \frac{6}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|$. Then, we get:

$$\begin{aligned}
|\Delta A_kB_kC_k| &= |\Delta A_{k-1}B_{k-1}C_{k-1}| - |\Delta A_{k-1}B_kC_k| - |\Delta B_{k-1}C_kA_k| - |\Delta C_{k-1}A_kB_k| \\
&= \left(1 - 3\cdot\frac{6}{49}\right)|\Delta A_{k-1}B_{k-1}C_{k-1}| \\
&= \frac{31}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|
\end{aligned}$$

Therefore, in our sequence of triangles, the ratio between two consecutive areas is $\frac{31}{49}$. Finally, since the area of our initial triangle $A_0B_0C_0$ is 400, our infinite geometric series is:

$$\begin{aligned}
400\left(1 - \frac{31}{49} + \left(\frac{31}{49}\right)^2 - \left(\frac{31}{49}\right)^3 + \cdots\right) &= 400\left(1 + \left(-\frac{31}{49}\right) + \left(-\frac{31}{49}\right)^2 + \cdots\right) \\
&= \frac{400}{1-(-\frac{31}{49})} \\
&= \frac{400}{\frac{80}{49}} \\
&= 245
\end{aligned}$$

$\therefore$ A total area of $\boxed{245}$ square units is contained inside an odd number of triangles.

6. For all integers $1 \leq n \leq 18$, define $f(n)$ to be the smallest positive integer k such that nk is congruent to either 1 or 18 modulo 19. Compute:

$$\sum_{n=1}^{18} f(n)$$

Solution:

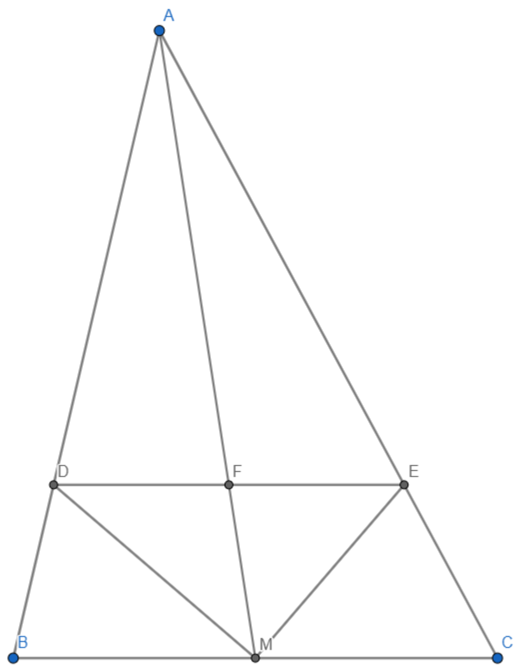
By multiplicative inverses, we can permute the integers $1, 2, 3, \ldots, 18$ as $b_1, b_2, b_3, \ldots, b_{18}$ such that $b_n \cdot n \equiv 1 \pmod{19}$ for all $1 \leq n \leq 18$. Then, $b_n \cdot (-n) \equiv -1 \equiv 18 \pmod{19}$, so $f(b_n) = \min(n, 19-n)$. Clearly, $f(b_n) = n$ for $1 \leq n \leq 9$, and $f(b_n) = 19 - n$ for $10 \leq n \leq 18$, so:

$$\begin{aligned}\sum_{n=1}^{18} f(n) &= \sum_{n=1}^{18} f(b_n) \\ &= (1 + 2 + \cdots + 9) + (9 + 8 + \cdots + 1) \\ &= 2 \cdot \frac{9 \cdot 10}{2} \\ &= 90\end{aligned}$$

$\therefore$ The required sum is $\boxed{90}$.

7. Let ABC be a triangle with $AB = 7$, $BC = 4$, and $CA = 9$. Let M be the midpoint of BC. Let the internal angle bisector of $\angle AMB$ intersect AB at D, and let the internal angle bisector of $\angle AMC$ intersect AC at E. Finally, let F be the midpoint of DE. Find AF.

Solution:



We first calculate AM. Set $AM = x$, and note that $BM = CM = 2$. Then, note that by Stewart's Theorem we have:

$$\begin{aligned} 4x^2 + 2 \cdot 4 \cdot 2 &= 7^2 \cdot 2 + 9^2 \cdot 2 \\ 4x^2 + 16 &= 98 + 162 \\ 4x^2 &= 98 + 162 - 16 \\ &= 244 \\ x &= \sqrt{61} \end{aligned}$$

Now, the next part of this problem relies on the following synthetic observations. Note that by the Angle Bisector Theorem, we have that:

$$\begin{aligned} \frac{AD}{DB} &= \frac{AM}{MB} \\ &= \frac{AM}{MC} \\ &= \frac{AE}{EC} \end{aligned}$$

Thus, we have that DE and BC are parallel, and in fact triangles ADE and ABC are homothetic with relation to A. This gives us that in fact F also lies on AM, and we have the relation:

$$\begin{aligned}\frac{AF}{FM} &= \frac{AD}{DB} \\ &= \frac{AM}{MB}\end{aligned}$$

Finally, we compute:

$$\begin{aligned}\frac{AF}{AM} &= \frac{AM}{AM+MB} \\ AF &= \frac{AM^2}{AM+MB} \\ &= \boxed{\frac{61}{2+\sqrt{61}}}\end{aligned}$$

NOTE: originally, in the original copy of this question we had swapped BC and CA: this would have led to the answer of $\frac{49}{32}$, which was the answer used during scoring at the time. Unfortunately, this text correction was not displayed on the in-contest problem sheet, and thus an incorrect answer was used for scoring.

8. We place a knight in a 3x3 grid such that it has at least one valid move (a knight, as defined in chess, can move two steps horizontally in either direction and one step vertically in either direction, or two steps vertically and one step horizontally). How many ways can we make 16 moves with it such that it ends up in the same square it starts on?

   Solution: The 8 possible squares the knight can be on form a cycle, so the answer is just $16C0 + 16C4 + 16C8 + 16C12 + 16C16$, as the number of moves counterclockwise in the cycle of length 8 must be a multiple of 4. This sum can be evaluated using roots of unity filter as the expression $\frac{2^{16}+(1+i)^{16}+(1-i)^{16}}{4}$ or long multiplication to be $\boxed{16512}$.

9. Michael repeatedly generates random integers between 1 and 20, inclusive, and writes them in order until the number 20 is generated. Compute the probability that he can then delete some written numbers (possibly none) until there are 10 distinct odd numbers followed by 10 distinct even numbers.

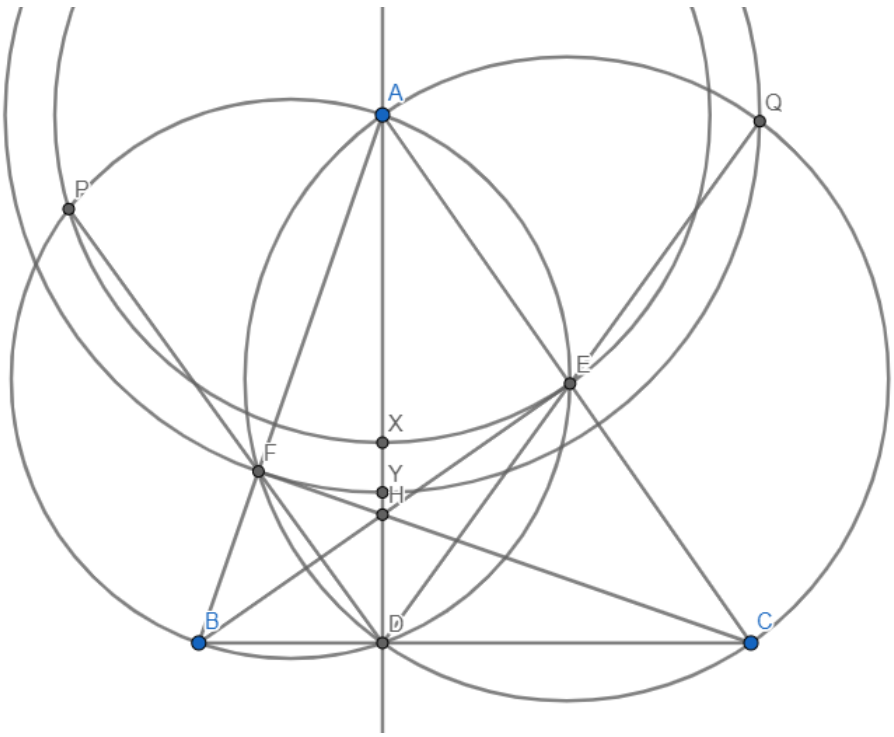
   Solution:

   First, Michael needs to generate all the odd numbers before the number 20. Basically, among the numbers $1, 3, 5, \ldots, 19, 20$, the number 20 should be generated last. Since this list contains 11 numbers, there is a $\frac{1}{11}$ chance that 20 is generated last.
   Next, after the last odd number is generated, Michael needs to generate all the even numbers, with 20 being generated last. Similarly to above, there are 10 even numbers, so there is a $\frac{1}{10}$ chance that 20 is generated last.

   $\therefore$ Overall, the required probability is $\frac{1}{11} \cdot \frac{1}{10} = \boxed{\frac{1}{110}}$.

10. Let ABC be a triangle, and let E and F be the feet of the altitudes from B to AC and C to AB respectively. Let D be the foot of the altitude from A to BC. Say line FD intersects the circumcircle of triangle ADB at $P \neq D$, and say ED intersects the circumcircle of triangle ADC at $Q \neq D$. Let X be the incenter of triangle DPE and Y be the incenter of triangle DQF. If $AB = 13$, $AC = 15$, and $BC = 14$, compute XY.

Solution:



First of all, consider the following lemma:

LEMMA: Let ABC be a triangle, I its incenter, and M the midpoint of arc BC not containing A. Then M is the circumcenter of triangle BIC.

This lemma is known as the Incenter-Excenter lemma: the proof is angle chasing, and can be found online.

Now, a few synthetic observations. First of all, note that quadrilaterals $BFEC$, $AEDB$, and $AFDC$ are cyclic. If we let H be the orthocenter, then we have that $BFHD$ and $CEHD$ are also cyclic. This gives us that:

$$\begin{aligned} \angle FDA &= \angle FBG \\ &= \angle ECG \\ &= \angle EDA \end{aligned}$$

In fact, this gives that $\angle PDA = \angle EDA$, which means that A is the midpoint of arc PE on the circumcircle of PDE not including D. In addition, since DA bisects $\angle PDE$ we have that X lies on DA, and by the incenter-excenter lemma we have that $AX = AE$. Similarly, we find that Y lies on

AD and $AY = AF$, giving us that:

$$\begin{aligned} XY &= AY - AX \\ &= AF - AE \\ &= AC \cos \angle BAC - AB \cos \angle BAC \\ &= 2 \cos \angle BAC \end{aligned}$$

We now calculate this value. Applying Heron's Formula gives us that the area of ABC is 84, but this is also equal to $\frac{1}{2} AB \cdot AC \cdot \sin \angle BAC$. This gives us that $\sin \angle BAC = \frac{168}{195}$, which finally gives us that:

$$\begin{aligned} XY &= 2 \cos \angle BAC \\ &= \frac{2\sqrt{195^2 - 168^2}}{195} \\ &= \boxed{\frac{198}{195}} \end{aligned}$$