# UTMC Senior Individual Round Solutions

UTMC Committee

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1. What is the maximum value of $a^{b^{c^d}}$, given that $a, b, c, d$ are all distinct integers chosen from the set $(0, 1, 2, 9)$?

   *Solution:*

   First, we obviously want $a > 1$, so $a = 2$ or 9. It is easy to see that in each case, the maximum possible value of the exponent $b^{c^d}$ is $9^{1^0} = 9$ or $2^{1^0} = 2$, respectively. This gives us maximum values of $2^9 = 512$ and $9^2 = 81$, respectively. The larger of these two values is $\boxed{512}$.

2. Define the operation $\diamond$ such that:

$$a \diamond b = ab - 9a + 9$$

Compute:

$$(\dots((1 \diamond 2) \diamond 3) \dots \diamond 9) \diamond 10$$

*Solution:*

The key idea of this problem is that $a \diamond 9 = 9a - 9a + 9 = 9$. This reduces the given expression to

$$9 \diamond 10 = 9 \cdot 10 - 9 \cdot 9 + 9 = \boxed{18}$$

3. Let $d$ be a randomly selected positive (not necessarily proper) divisor of $6^6$. Compute the expected value of the number of positive divisors of $d$.

*Solution:*

Since $6^6 = 2^6 \cdot 3^6$, the prime factorization of $d$ must be in the form $2^m \cdot 3^n$, where $m$ and $n$ are nonnegative integers such that $m \leq 6$ and $n \leq 6$. Then, $d$ has $(m+1)(n+1)$ positive divisors. Since there are $(6+1)(6+1) = 49$ possible values of $d$, the expected number of positive divisors of $d$ is:

$$\begin{aligned}
\frac{1}{49}\sum_{m=0}^{6}\sum_{n=0}^{6}(m+1)(n+1) &= \frac{1}{49}\left(\sum_{m=0}^{6}(m+1)\right)\sum_{n=0}^{6}(n+1) \\
&= \frac{1}{49}(1+2+\cdots+7)(1+2+\cdots+7) \\
&= \frac{1}{49}\cdot\frac{7\cdot 8}{2}\cdot\frac{7\cdot 8}{2} \\
&= \frac{8}{2}\cdot\frac{8}{2} \\
&= 16
\end{aligned}$$

$\therefore$ The expected number of positive divisors of $d$ is $\boxed{16}$.

4. For non-zero real numbers m and n, define the operation ★ as $m \bigstar n = \frac{1}{m} + \frac{1}{n} - \frac{1}{mn}$. Compute the value of

$$4\bigstar(5\bigstar(\dots(2018\bigstar 2019)\dots))$$

*Solution:*

We work backwards starting from the inside bracket and moving out. For the first term, we find $\frac{1}{2018} + \frac{1}{2019} - \frac{1}{2018*2019} = \frac{2}{2019}$. The second term is $\frac{1}{2017} + \frac{2019}{2} - \frac{2019}{2*2017} = 1009$. Now that we've established out base cases, we can show that for all $n$, $\frac{1}{2020-2n} + \frac{1}{\frac{2020-n}{n}} - \frac{1}{(2020-2n)\frac{2020-n}{n}} = \frac{n+1}{2020-n}$ and that $\frac{1}{2020-2n-1} + \frac{1}{\frac{n+1}{2020-n}} - \frac{1}{(2020-2n-1)\frac{n+1}{2020-n}} = \frac{2020-n-1}{n+1}$. Thus, it's clear by induction that for all even integers $2020-2n$ on the left side, the final result would be $\frac{n+1}{2020-n}$. Thus, for $2020-2n=4$, we have that $n=1008$ so our final answer is $\boxed{\frac{1009}{1012}}$

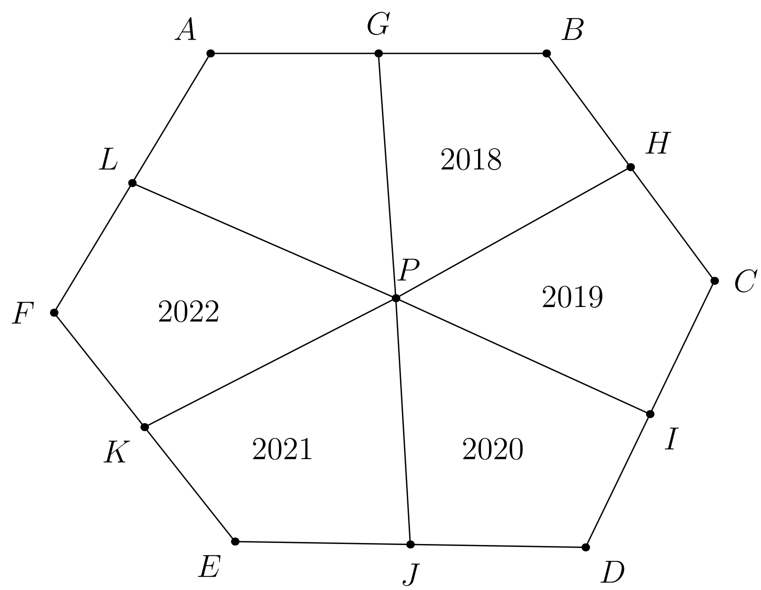
5. Given that there exists four primes for which the sum of any three is a prime number, find the minimum possible value of the smallest prime out of the four.

   *Solution:*

   If the smallest prime is 3, then the other three primes must be either 1 or 2 modulo 3. We find that for any combination of other primes, we can always find three of these primes that sum to a multiple of 3, while this sum is obviously $> 3$ so it cannot be prime. Thus, the smallest prime must be 5. The construction for this is the primes $5, 7, 17, 19$.

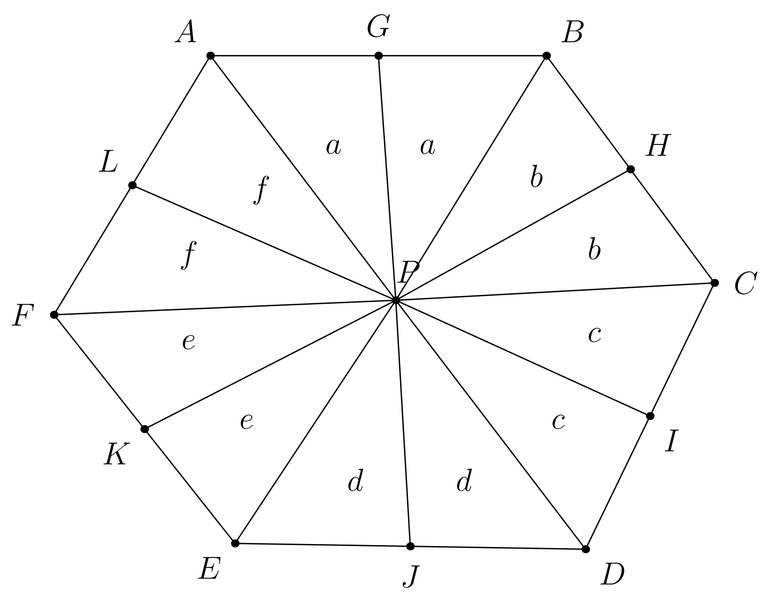
6. Let $ABCDEF$ be an convex hexagon. Let points $G$, $H$, $I$, $J$, $K$, and $L$ be the midpoints of $AB$, $BC$, $CD$, $DE$, $EF$, and $FA$, respectively. Let point $P$ be a point on the interior of the hexagon such that the areas of quadrilaterals $BGPH$, $CHPI$, $DIPJ$, $EJPK$, and $FKPL$ are 2018, 2019, 2020, 2021, and 2022, respectively. Compute the area of hexagon $ABCDEF$.

*Solution:*




Note that since $G$ is the midpoint of $AB$, $[AGP] = [BGP]$. Then, we also get $[BPH] = [CPH], [CPI] = [DPI], [DPJ] = [EPJ], [EPK] = [FPK]$, and $[FPL] = [APL]$.

Connect $PA, PB, PC, PD, PE, PF$ as follows and label the areas.




We wish to find $[ALPG]$ to complete the areas of the hexagon.

$[ALPG] = a+f = (a+b)-(b+c)+(c+d)-(d+e)+(e+f) = 2018-2019+2020-2021+2022 = 2020$

Therefore, the area of the hexagon is $2018 + 2019 + 2020 + 2021 + 2022 + 2020 = \boxed{12120}$

7. The company MMM (Make More Money) made a whopping profit of \$20 last year, and the $N$ owners are splitting the money using the following process. The youngest owner proposes a plan where each owner receives a positive integer number of dollars, such that the youngest owner maximizes their own profit while still satisfying the other owners. After the split, each owner votes. (Assume that all owners have different ages.) If at most 1 owner votes against the plan, it passes; otherwise, the youngest owner gets fired and receives \$0, and the process repeats for the remaining $N-1$ owners. Each owner tries to maximize the amount of money they receive; if there are multiple possible moves that do this, they try to fire as many owners as possible. Assume that the owners do not collude, but that they act optimally otherwise. Given that $N = 4$, compute the amount of money that the youngest owner receives.

   *Solution:*

   First, if $N = 2$, then the youngest owner maximizes their amount of money by giving the oldest owner only \$1 and keeping the rest of the money. Even if the oldest owner votes against this plan, the youngest owner does not get fired.
   Then, if $N = 3$, the youngest owner can give the oldest owner only \$2, then the oldest owner will not vote against the plan (otherwise, they will only earn \$1.) The youngest owner can then receive \$17, leaving only \$1 for the middle owner.
   Similarly, if $N = 4$, the youngest owner can give \$3 and \$2 to the two oldest owners, and then they will not vote against this plan. The second-youngest owner must receive \$1, then the youngest owner happily earns \$14.
   $\therefore$ When $N = 4$, the youngest owner earns $\boxed{\$14}$.

8. Let $ABC$ be a triangle satisfying $AB = 3$, $AC = 5$, $BC = 7$. Let $A'$ be the reflection of $A$ across $BC$, and let the tangents to the circumcircle of triangle $ABC$ from points $B$ and $C$ intersect at a point $K$. Find $A'K$.

   *Solution:*

   The key idea behind this problem is to note that if we let $\cos \angle BAC = x$, then we have by cosine law that:

$$\begin{aligned} 7^2 &= 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot x \\ 49 &= 34 - 30x \\ x &= -\frac{1}{2} \\ \angle BAC &= 120^\circ \end{aligned}$$

   This gives us that $\angle BA'C = 120^\circ$, and $\angle KBC = \angle KCB = 180^\circ - \angle BAC = 60^\circ$. This in fact indicates that $KBC$ is equilateral, and now $\angle BKC = 60^\circ = 180^\circ - \angle BA'C$ gives us that in fact $BKCA'$ is cyclic.

   Now, we apply Ptolemy to $BKCA'$ (or the fabled van Schooten theorem) to find that:

$$\begin{aligned} BK \cdot A'C + CK \cdot A'B &= BC \cdot A'K \\ A'K &= A'C + A'B \\ &= AC + AB \\ &= 5 + 3 \\ &= \boxed{8} \end{aligned}$$

9. Michael is playing a game with his 4 friends to split a single cake. To begin, each player flips one coin. Then, all people who flip "Heads" share the cake equally. If everyone flips "Tails", the cake is thrown out and nobody gets to eat it. What is the expected amount of cake that Michael will get?

   *Solution:*

   By symmetry, each of the 5 players will get an equal expected amount of cake. Thus, we simply need to find the expected amount of cake that is given out, then divide by 5.
   No cake is given out if everyone flips "Tails", and the entire cake is given out otherwise. There is only a $\frac{1}{2^5} = \frac{1}{32}$ chance that everyone flips "Tails", so the expected amount of cake is $1 - \frac{1}{32} = \frac{31}{32}$.
   $\therefore$ The expected amount of cake received by Michael is $\frac{31}{32} \cdot \frac{1}{5} = \boxed{\frac{31}{160}}$.

10. Let $x$ be a real number such that $\log_4(x), \log_8(4x),$ and $2020 + \log_{32}(x)$ are three consecutive terms in a geometric sequence (in that order). The product of all possible values of $x$ can be written in the form $2^n$ for some real number $n$. Compute $n$.

*Solution:*

To be consecutive terms in a geometric sequence, $\log_4(x), \log_8(4x),$ and $2020 + \log_{32}(x)$ must satisfy the relation

$$\log_4(x)\,(2020 + \log_{32}(x)) = (\log_8(4x))^2$$

Converting all logs to base 2, we get

$$\frac{1}{2}\log_2(x)\left(2020 + \frac{1}{5}\log_2(x)\right) = \left(\frac{1}{3}\log_2(4x)\right)^2$$

Setting $y = \log_2(x)$, this simplifies to

$$\frac{y}{2}\left(2020 + \frac{y}{5}\right) = \left(\frac{y}{3} + \frac{2}{3}\right)^2$$

Expand and rearrange to get

$$\frac{y^2}{10} + 1010y = \frac{y^2}{9} + \frac{4y}{9} + \frac{4}{9}$$

$$y^2 - 90860y + 40 = 0$$

Since the discriminant of this is clearly larger than 0, there are 2 real solutions to the quadratic. Let these solutions be $y_1$ and $y_2$.

Also since $y = \log_2(x)$, we have $x = 2^y$.

Then, the product of the solutions is $x_1 x_2 = 2^{y_1} \cdot 2^{y_2} = 2^{y_1+y_2}$.

By Vieta's this is $2^{90860}$ and our answer is $\boxed{90860}$

11. $a$ and $b$ are one digit positive integers in base $k$. If $2a \cdot b$ represented in base $k$ forms the two digit number $ab$, find the number of possible values of integers $k$ from 2 to 2020 inclusive. Here, we say that the two digit number $ab$ is the number formed by sticking the digits $a$ and $b$ in base $k$ together in that order: for example, if $k = 10$, $a = 3$, and $b = 5$, then this number would be 35.

*Solution*

We just have to solve for $2ab = ka + b$ for when $a, b$ are between 1 and $k - 1$. For all $k$ that aren't a power of two, it has an odd prime divisor $p \geq 3$. Thus, notice that the above equation simplifies to $b = \frac{ka}{2a-1} < k$ if $a \geq 2$. Thus, we can simply choose $a$ such that $2a - 1 = p$ so that $a$ and $b$ are both positive integers $< k$. Now, if $k$ is a power of 2, then we cannot have $2a - 1$ divides $k$, so $b$ will always be a positive integer multiple of $k$, so $b \geq k$ which fails. Thus, we simply count the number of non-powers of 2 from 2 to 2020, which is $2020 - 11 = \boxed{2009}$ as we have 11 powers of two that are less than 2020, including 1, and we have 2020 integers from 1 to 2020.

12. Let $ABC$ be a triangle with $AB = 5$, $AC = 12$, and $BC = 13$. Let the incircle of $ABC$ touch $AB$ and $AC$ at $F$ and $E$ respectively, and let the line $EF$ intersect the circumcircle of triangle $ABC$ at two points $P$ and $Q$. Compute the length of $PQ$.

*Solution 1:*

We first present a boring power of a point solution. We WLOG assume that $E$ is closer to $P$ than $F$ is, on the segment $PQ$. Denote $PE = x$, $QF = y$. We first calculate:

$$\begin{aligned} AE &= \frac{AB + AC - BC}{2} \\ &= \frac{5 + 12 - 13}{2} \\ &= 2 \\ EC &= 12 - 2 \\ &= 10 \\ FB &= 5 - 2 \\ &= 3 \\ EF &= AE\sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

We now perform power of a point from both $E$ and $F$:

$$\begin{aligned} x(y + 2\sqrt{2}) &= 20 \\ &= xy + 2x\sqrt{2} \\ (x + 2\sqrt{2})y &= 6 \\ &= xy + 2y\sqrt{2} \\ 2\sqrt{2}(x - y) &= 14 \\ x - y &= \frac{7}{\sqrt{2}} \end{aligned}$$

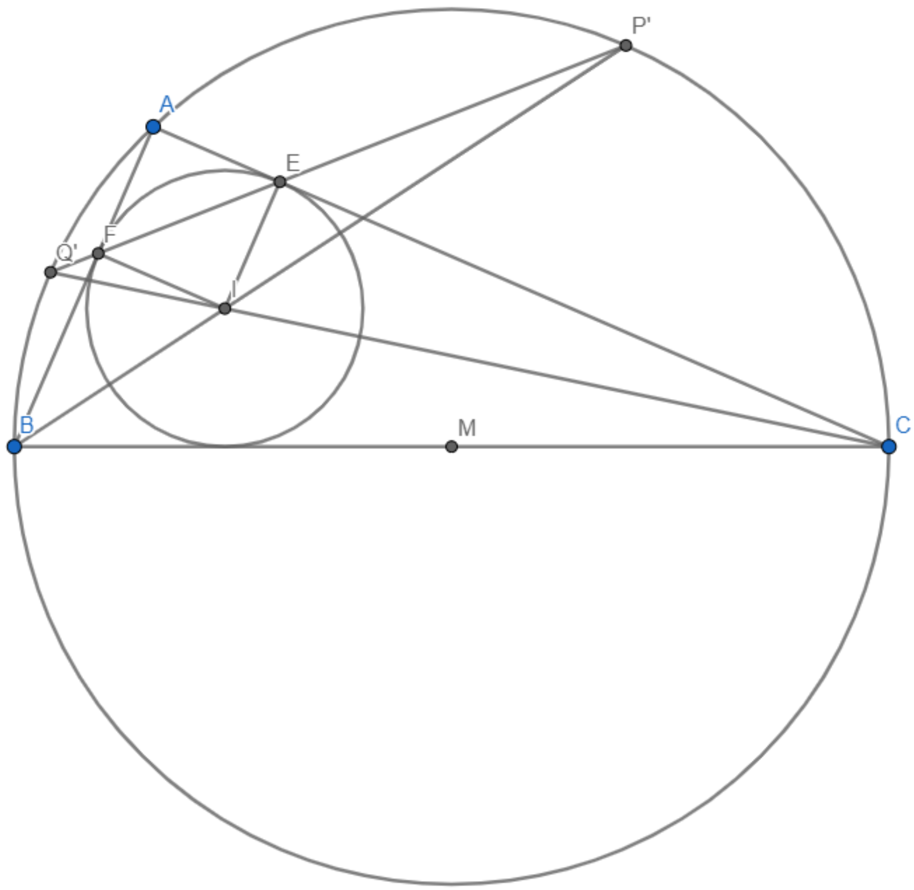
Plugging this back into the second relation, we find that:

$$\begin{aligned} y(y + \frac{7}{\sqrt{2}} + 2\sqrt{2}) &= y(y + \frac{11}{\sqrt{2}}) \\ &= 6 \\ y^2 + \frac{11}{\sqrt{2}}y - 6 &= 0 \\ (y + \frac{12}{\sqrt{2}})(y - \frac{1}{\sqrt{2}}) &= 0 \end{aligned}$$

Since $y$ is positive, this gives us that $y = \frac{1}{\sqrt{2}}$, and in turn we find that $x = \frac{8}{\sqrt{2}}$. This gives us that:

$$\begin{aligned} PQ &= QF + FE + EP \\ &= \boxed{\frac{13}{\sqrt{2}}} \end{aligned}$$

*Solution 2:*




We now present a slick synthetic solution that relies on basically no calculation.

Let $P'$ be the midpoint of the arc $AC$ not including $B$. We claim that $P'$ lies on $EF$. Throughout this solution we will denote $I$ as the incenter of triangle $ABC$, and $M$ as the circumcenter of $ABC$ (and the midpoint of $BC$). Note that it suffices to prove that $\angle FEA = \angle P'EC$

First of all, note that $\angle BP'C = 90° = \angle IEC$. This gives us that quadrilateral $IEPC$ is cyclic. Now, by the incenter-excenter lemma, we have that $P'I = P'C$, and since $\angle IP'C = 90°$ we have that $\angle PI'C = 45°$. This gives us that $\angle P'EC = \angle P'IC = 45°$. However, since we have that $\angle AEF = 45°$ as well (as $AEIF$ is a square), we find that $P'$ lies on $FE$, as desired.

Now, repeating the same argument for the midpoint of arc $AB$ not containing $C$ (which we now denote as $Q'$ gives us that $P'Q' = PQ$. However, we have that the circumradius of $ABC$ is equal to half the length of the hypotenuse, or $\frac{13}{2}$, and since $\angle P'MQ' = \frac{1}{2}\angle BMC = 90°$, we have that $P'Q' = P'M \cdot \sqrt{2} = \boxed{\frac{13}{\sqrt{2}}}$, as desired.

**REMARK:** The lemma presented within the solution is a special case of the so-called *Iran incenter lemma*. For a generalized version of this lemma, look at USAJMO 2014 6 part a.