UTMC Marathon Round Solutions

UTMC Committee

March 2020

Set 1

1. Compute the number of positive integers k such that $\frac{2020-k}{k}$ is also a positive integer.

   Solution:

   We can separate the fraction to $\frac{2020}{k} - 1$ so we only need to find the number of factors of 2020. Since 2020 prime factorizes as $2^2 \cdot 5 \cdot 101$, its number of positive factors is $(2+1)(1+1)(1+1) = 12$. However, $\frac{2020-k}{k}$ must also be positive, which means that we must exclude $k = 2020$. Thus, $\boxed{11}$ values of k satisfy the desired condition.

2. Solve for x:
$$(2+0+2+0)x = 2020$$

   Solution:

   This is just $4x = 2020$ or $x = \boxed{505}$.

3. Two nonzero real numbers a and b satisfy $\frac{a}{b} = 2020$. Compute all possible values of $\frac{\min(a,b)}{\max(a,b)}$.

   Solution: We can split into two cases, when a is positive or negative. It's $\frac{1}{2020}$ if $a > 0$, and 2020 if $a < 0$, so the answer is $\boxed{\{\frac{1}{2020}, 2020\}}$

Set 2

1. The Kansas City Chiefs played a football game against the Toronto Pigeons! Whenever the Chiefs started a drive, the drive ended in one of two ways:

   - Regular touchdown, worth 7 points.
   - Touchdown with two-point conversion, worth 8 points.

   Overall, the Chiefs scored T touchdowns and earned 100 points from them (while the Pigeons did not score at all). Compute the number of possible values of T.

   Solution:
   Suppose the Chiefs scored x regular touchdowns and y two-point conversions. Then, the total number of touchdowns is $x + y = T$, and since the Chiefs scored 100 points from them, we get $7x + 8y = 100$. Subtracting the second equation from 8 times the first equation, we get $x = 8T - 100$. Substituting this into $x + y = T$, we also get $y + 8T - 100 = T$, so $y = -7T + 100$.
   Clearly, $x \geq 0$, so $8T - 100 \geq 0$, from which we get $T \geq \frac{100}{8}$. Since T is an integer, it follows that $T \geq 13$. Also, $y \geq 0$, so $-7T + 100 \geq 0$, from which we get $T \leq \frac{100}{7}$. Since T is an integer, it follows that $T \leq 14$.
   $\therefore$ The value of T is either 13 or 14, so there are $\boxed{2}$ possible values of T.

2. Ana and Banana are setting problems for a math contest. Working alone, Ana can create one problem in 48 minutes. However, when she works together with Banana, they both work at twice their regular speeds. This allows them to create one problem in only 8 minutes. Given that both Ana and Banana create problems at constant rates, how many minutes does Banana when working alone need to create one problem?

   Solution:
   Working alone, Ana creates $\frac{1}{48}$ of a problem per minute. Then, when she works together with Banana, her pace doubles, and she can create $2 \cdot \frac{1}{48} = \frac{1}{24}$ of a problem per minute. Together, this team creates $\frac{1}{8}$ of a problem per minute, which means that Banana contributes $\frac{1}{8} - \frac{1}{24} = \frac{2}{24} = \frac{1}{12}$ of a problem each minute. However, when she works alone, she only creates $\frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$ of a problem per minute.
   $\therefore$ Working alone, Banana can come up with one problem in $\boxed{24}$ minutes.

3. A number from 1 to 200, inclusive, is chosen. What is the probability that it can be expressed as a difference of two non-negative squares?

   Solution:
   Note that since all perfect squares are 0 or $1 (\mod 4)$, we have that if an integer is expressible as a difference of two perfect squares then it cannot be equivalent to $2 \mod 4$. It turns out that all other integers work: odd numbers can be covered by consecutive squares ($1^2 - 0^2 = 1$, $2^2 - 1^2 = 3$, and so on), and any multiple of 4 (which we will express as $4k$ for some integer k) can be expressed as $(4k^2 + 1)^2 - (4k^2 - 1)^2$. Thus, since $\frac{3}{4}$ of the integers between 1 and 200 inclusive are not equivalent to $2 (\mod 4)$, our answer is $\boxed{\frac{3}{4}}$

Set 3

1. Let ABC be a triangle. Let M be the midpoint of BC, and let P and Q be on AC and AB respectively such that $\dfrac{AP}{PC} = \dfrac{BQ}{QA} = 2$. If D is the midpoint of BP, E is the midpoint of CQ, F is the midpoint of MP, and G is the midpoint of QM, find the ratio between the areas of quadrilateral $FEGD$ and triangle ABC.

   Solution:

   Let the distance between DF and EG be h, let the distance from A to BC be H, and let $BC = a$.Note that by similarity, EG and DF are both parallel to BC. In addition, note that the distance from Q to BC is $\frac{2H}{3}$, and the distance from P to BC is $\frac{H}{3}$. This gives us that the distance from E to BC is $\frac{H}{3}$, and the distance from F to BC is $\frac{H}{6}$, so $h = \frac{H}{3} - \frac{H}{6} = \frac{H}{6}$.

   Now, note that by similarity $DF = EG = \frac{a}{2}$. This gives us that the area of $DFEG$ is $\frac{a}{4} \cdot \frac{H}{6} = \frac{aH}{24}$. Since the area of ABC is $\frac{1}{2}aH$, we have that the ratio of the areas is $\boxed{\dfrac{1}{12}}$

2. Let $a_1, a_2, \ldots$ be a sequence recursively defined as:

   (a) $a_1 = 1$

   (b) $a_2 = 3$

   (c) $a_k = 4a_{k-1} - 4a_{k-2} + 1$

   Compute a_{11}.

   Solution:

   Define $b_k = a_k - 2a_{k-1} + 1$. Then the third condition rearranges to give $b_k = 2b_{k-1}$. Since $b_2 = 2$, we have by induction that $b_k = 2^{k-1}$. This gives us that:

$$\begin{aligned} a_{11} &= b_{11} - 1 + 2a_{10} \\ &= b_{11} - 1 + 2b_{10} - 2 + 4a_9 \\ \ldots \\ &= b_{11} - 1 + 2b_{10} - 2 \ldots + 512b_2 - 512 + 2^{10}a_1 \\ &= b_{11} + 2b_{10} + \ldots 512b_2 + 1 \\ &= 10 \cdot 1024 + 1 \\ &= 10241 \end{aligned}$$

   REMARK: In contest the answer to this question was given as 8193 erroneously. We apologize for the mistake.

3. Let k be a positive integer in base 4, and let $\#\#\#_4$ denote a numeral in base 4. We wish to reduce k to 0 by repeating the following procedure: Select any two nonzero digits of k (in base 4), and reduce each of those digits by 1. For how many positive integers $k \leq 1000_4$ is this possible? (For example, for $k = 222_4$, we can do $2\underline{22} \to \underline{2}1\underline{1} \to \underline{11}0 \to 0$.)

   Solution:
   Since the procedure decreases the sum of the digits by 2 for every step, the sum of the digits of k must be even. Additionally, the largest digit of k must be less than or equal to the sum of the other digits, or else the largest digit cannot be reduced to 0. Then, if these two conditions are satisfied, it is always possible to reduce k to 0 by repeatedly applying the procedure to the two largest digits.
   Now, we pair up the positive integers up to 1000_4 as:

$$\{2_4, 3_4\}, \{10_4, 11_4\}, \{12_4, 13_4\}, \ldots, \{332_4, 333_4\}, \{1000_4, 1_4\}$$

   such that each pair has exactly one integer with an even digit sum, except the last pair. Thus, $\frac{64}{2} - 1 = 31$ of the integers have an even digit sum.

Then, we subtract the cases where the largest digit of k is too large and k has an even digit sum. Keeping in mind that $k \leq 1000_4$, here is a table of the number of subtracted cases for each combination of digits.

Digits	Number of cases
0, 0, 2	3
0, 1, 3	6
Total:	9

$\therefore$ A total of $31 - 9 = \boxed{22}$ values of k can be reduced to 0.

Set 4

1. Consider an isosceles right triangle AOB such that $AO = BO$. Say A, O, and B all lie on a semicircle such that AB is the base of this semicircle. Consider points C and D on the semicircle such that AC bisects $\angle OAB$ and BD bisects $\angle OBA$. Finally, denote X as the intersection of AC and BD. Find $\angle CXD$.

   Solution:

   We have:

$$\begin{aligned}\angle CXD &= \angle AXB \\ &= 180^\circ - \angle XAB - \angle XBA \\ &= 180^\circ - \frac{1}{2}(\angle OAB + \angle OBA \\ &= 180^\circ - 45^\circ \\ &= \boxed{135^\circ}\end{aligned}$$

2. Let $ABCD$ be a rectangle such that $AB = 3$ and $BC = 4$. Consider a point P on line BD such that the centroid of APC lies on BC Find the area of triangle APC.

   Solution:

   Let M be the intersection of AC and BD. Note that M is the midpoint of AC, and thus the centroid of APC must lie on MP and BC. The intersection of these two lines is B, so the centroid of APC must be B.

   Now, we have that $\frac{PM}{BM} = 3$, so the area of APC is three times the area of ABC. Thus, the area of APC is:

$$\frac{1}{2} \cdot 3 \cdot 4 \cdot 3 = \boxed{18}$$

3. Michael is trying to roll a 6 with a standard six-sided die. However, before each roll, there is an independent probability of p that he gets distracted and never rolls the die again. If he has a $\frac{5}{6}$ chance of eventually rolling a 6, compute the value of p.

   Solution:
   There is a probability of $(1 - p)$ that Michael actually rolls the die. If he does, he has a $\frac{1}{6}$ chance of immediately rolling a 6. He also has a $\frac{5}{6}$ chance of rolling something else, then he again has a $\frac{5}{6}$ chance of eventually rolling a 6. Thus, $\frac{5}{6} = (1 - p)(\frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6}) = \frac{31}{36}(1 - p)$.
   $\therefore 1 - p = \frac{30}{31}$, so $p = \boxed{\frac{1}{31}}$.

Set 5

1. Consider a triangle ABC where $AB < AC$. Let I be its incenter and let K be a point on the circumcircle of ABC such that KB is perpendicular to AI. Let KC intersect the circumcircle of BIC again at M. If M is the A-excenter and $\angle ABC = 40°$, find $\angle ACB$.

   Solution:

   Note that by the incenter-excenter lemma we have that $BIDCM$ is cyclic. Now, we have:

$$\begin{aligned}
\angle DBM &= \angle KCD \\
&= \angle ABK \\
&= 90° - \angle BAI \\
&= 90° - \frac{1}{2}\angle BAC \\
\angle IBD &= \angle ICA \\
&= \frac{1}{2}\angle ACB \\
90° &= \angle IBM \\
&= \angle IBD + \angle KBM \\
&= 90° + \frac{1}{2}(\angle ACB - \angle BAC) \\
\angle ACB &= \angle BAC
\end{aligned}$$

   This gives us that $\angle BCA = 90° - \frac{1}{2}\angle ABC = \boxed{70°}$

2. Simhon wins a game with probability p between 0 and 1, inclusive. If he wins, he will get $(3 - 3p)$ dollars, but if he loses he will lose $(p + 1)$ dollars. Compute the value of p that maximizes the expected value of his profit.

   Solution:
   The probability that Simhon will win $(3 - 3p)$ dollars is p, and the probability that he will lose $(p+1)$ dollars is $(1 - p)$. Then, his expected profit is:

$$\begin{aligned}
&(3 - 3p)p - (p + 1)(1 - p) \\
&= (-3p^2 + 3p) - (1 - p^2) \\
&= -2p^2 + 3p - 1
\end{aligned}$$

   This is a quadratic function that opens down. Thus, this function is maximized at the x-coordinate of the vertex, which is $-\dfrac{b}{2a} = -\dfrac{3}{2(-2)} = \dfrac{3}{4}$.

   $\therefore$ The probability p that maximizes Simhon's profit is $\boxed{\dfrac{3}{4}}$ or $\boxed{75\%}$.

3. A *harmonic sequence* $a_1, a_2, \ldots$ is defined such that

$$\begin{aligned}
\frac{a_2 - a_1}{a_1} &= \frac{a_3 - a_2}{a_3} \\
\frac{a_3 - a_2}{a_2} &= \frac{a_4 - a_3}{a_4}
\end{aligned}$$

   and so on, where all members of the sequence are nonnegative.

   Consider a harmonic sequence that satisfies that a_1 is an integer, $a_2 = 10$, and the sequence is strictly increasing (so no two terms are equal). Given that this harmonic sequence is not infinitely long, find the largest possible length of the sequence.

Solution:

Note that the condition rearranges as $\frac{1}{a_1}, \frac{1}{a_2}, \ldots$ being an arithmetic sequence. Now, let $\frac{1}{a_1} - \frac{1}{a_2} = d$. Note that the length of the sequences is maximized when d is minimized (as each term after $\frac{1}{a_2}$ is d less than the previous term but is positive. Since d is at least $\frac{1}{9} - \frac{1}{10} = \frac{1}{90}$, we can set d to be equal to this value to give us that $\frac{1}{a_10} > 0$, but $\frac{1}{a_11} = 0$, a contradiction. Since this value of d maximizes the length of this sequence, we have that the maximum length possible is $\boxed{10}$

Set 6

1. You receive the following problem during a UTMC Individual round:

> Let r_1 and r_2 be the two distinct nonzero roots of the cubic equation $Q(x) = x^3 - 20yx^2 + 15yx = 0$. Compute the value of $\frac{1}{r_1} + \frac{1}{r_2}$.

   Interestingly, the question never gives you the value of y. Compute the value of $\frac{1}{r_1} + \frac{1}{r_2}$ anyway, and express it as a real number not in terms of y.

   Solution:
   First, $Q(x) = 0$ can be factored as $x(x^2 - 20yx + 15y) = 0$. Since r_1 and r_2 are nonzero roots, they must be the distinct roots of $x^2 - 20yx + 15y = 0$.
   Now, by Vieta's Formulas, $r_1 + r_2 = 20y$ and $r_1 r_2 = 15y$. Then, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{20y}{15y} = \frac{4}{3}$.
   $\therefore$ No matter what the value of y is, the value of $\frac{1}{r_1} + \frac{1}{r_2}$ is $\boxed{\frac{4}{3}}$.

2. Let A be a point on a circle, and let B and C be two different points on this circle such that $AB = AC \neq 0$. Consider two points D and E on the circumcircle of ABC such that AD and BE are parallel and AE and CD are parallel. Finally, denote X as the intersection of BD and CE. If $AX = DE$, and A, B, E, D, and C lie on the circle in this order, compute $\angle DAE$.

   Solution:

   Since $AX = DE = AB$ by the parallel condition, we have that the circumcenter of BXC is A. Also, note that since $AB = AC$, the entire diagram is symmetric with relation to AX. Now, we have the following angle equalities:

$$\begin{aligned}
\angle BXC &= 180^\circ - \frac{1}{2}\angle BAC \\
\angle BXC &= \angle BDC + \angle ECD \\
&= 180^\circ - \angle BAC + \angle EAD \\
\angle EAD &= \frac{1}{2}\angle BAC
\end{aligned}$$

   By the symmetry conditio nwe find that $\angle EAX = \frac{1}{2}\angle BAX$, so since $AB = AX$ we have that B and X are reflections with relation to AE. Similarly, we have that C and X are reflections with relation to AD. However, the only such point satisfying both conditions is known to be the orthocenter of ADE (one could rigorously prove this by proving the result for the orthocenter and then proving only one point satisfies the result).

   Now, since DB and AE are parallel but $ABED$ is an isoceles trapezoid, we have that $\angle DBE = 45^\circ$. Thus, we have that $\angle DAE = \angle DBE = 45^\circ$.

3. The UTMC Committee loves making problems, but such hard work must come at a cost! Quite literally, as the nth problem on the contest consumes $\frac{n^2+1}{n}$ dollars. If the committee begins with 2020 dollars, and cannot create a problem unless it can pay the full cost of producing it, what's the maximum number of problems that can they make?

   Solution:

Let S_n be the total number of dollars consumed for the first n problems. Then, we need to find the largest n such that $S_n \leq 2020$. The value of S_n is:

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \frac{i^2+1}{i} \\ &= \sum_{i=1}^{n} \left(i + \frac{1}{i}\right) \\ &= \frac{n(n+1)}{2} + \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

Since $\sum_{i=1}^{n} \frac{1}{i}$ is likely to be quite small, we can begin by finding the largest triangular number that is less than 2020. This turns out to be $\frac{63 \cdot 64}{2} = 2016$. Then, we make the following comparison between S_{63} and 2020:

$$\begin{aligned} S_{63} &= \frac{63 \cdot 64}{2} + \sum_{i=1}^{63} \frac{1}{i} \\ &= 2016 + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{63}\right) \\ &= 2016 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots + \left(\frac{1}{33} + \cdots + \frac{1}{63}\right) \\ &> 2016 + \frac{25}{12} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{32} + \cdots + \frac{1}{32}\right) + \left(\frac{1}{63} + \cdots + \frac{1}{63}\right) \\ &= 2016 + \frac{25}{12} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{31}{63} \\ &= 2016 + \frac{43}{12} + \frac{31}{63} \\ &= 2016 + \frac{1027}{252} \\ &> 2016 + 4 \\ &= 2020 \end{aligned}$$

Thus, S_{63} is too large, so we can check S_{62} next:

$$\begin{aligned} S_{62} &= \frac{62 \cdot 63}{2} + \sum_{i=1}^{62} \frac{1}{i} \\ &= 1953 + \sum_{i=1}^{62} \frac{1}{i} \\ &< 1953 + 62 \\ &= 2015 \\ &< 2020 \end{aligned}$$

$\therefore$ We find that S_{62} is small enough, so the UTMC Committee can afford to create $\boxed{62}$ problems.

Set 7

1. a, b, and c are the (possibly complex) roots of the polynomial

$$x^3 - 6x^2 + 12x - 9$$

If $|a| \geq |b| \geq |c|$, find $|a(b+c)|$.

Solution:

Note that the expression is equivalent to $(x-2)^3 - 1$ and thus its roots satisfy $(x-2)^3 = 1$. Solving this equation gives us the roots 3 and $\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$. 3 is greater in magnitude than the latter two roots, and thus $a = 3$. Finally, we can now evaluate $|a(b+c)|$ to get $\boxed{9}$

2. Let $f(n)$ denote the number of positive factors of an integer n. Find the number of integers n between 1 and 2019 inclusive such that $f(n)$ is an odd prime.

Solution:

$f(n)$ is the number of divisors of a number, an so it can only be an odd prime if n is a perfect even power of a prime. Thus, it suffices to check $f(n) = 3, 5, 7, 11$.

If $f(n) = 3$, we have that $n = p^2$ for some prime p, and the values $2^2, 3^2, 5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2,$ $37^2, 41^2, 43^2$ all work.

If $f(n) = 5$, then $n = p^4$ for some prime p, giving the solutions $n = 2^4, 3^4, 5^4$.

Similarly, $f(n) = 7$ gives $n = 2^6, 3^6$, and $f(n) = 11$ gives $n = 2^{10}$. Finally, counting over all the cases gives $\boxed{20}$ solutions in total.

3. Let ABC be a triangle with $AB = 32$, $AC = 50$, $BC = 45$. Let ω be a circle passing through A and tangent to BC at T, that intersects AB at D and AC at E. If $BD = CE$, compute BT.

Solution:

Power of a point from B gives us that $BT^2 = BD \cdot BA$. Power of a point from C gives us that $CT^2 = CE \cdot CA$. Thus, we have that:

$$\begin{aligned} \left(\frac{BT}{CT}\right)^2 &= \frac{BA}{CA} \\ &= \frac{16}{25} \\ \frac{BT}{CT} &= \frac{4}{5} \end{aligned}$$

Thus, $BC = \frac{9}{4}BT$, and thus $BT = \boxed{20}$

Set 8

1. Compute the maximum possible remainder when $86^n - 30^n + 21^n - 5^n$ is divided by 216 over all nonnegative integers n.

   Solution:
   First, we manually compute the remainders for $n = 0$, 1, and 2.
   For $n = 0$: $86^0 - 30^0 + 21^0 - 5^0 = 0 \equiv 0 \pmod{216}$.
   For $n = 1$: $86^1 - 30^1 + 21^1 - 5^1 = 72 \equiv 72 \pmod{216}$.
   For $n = 2$: $86^2 - 30^2 + 21^2 - 5^2 = 6912 \equiv 0 \pmod{216}$.
   Next, if $n \geq 3$, 86^n and 30^n are both divisible by 8 because 86 and 30 are both even. Also, since $21 \equiv 5 \pmod 8$, we get $21^n \equiv 5^n \pmod 8$, so $21^n - 5^n$ is divisible by 8. Thus, the entire expression $86^n - 30^n + 21^n - 5^n$ is divisible by 8. Similarly, 30^n, 21^n, and $86^n - 5^n$ are all divisible by 27 since $86 \equiv 5 \pmod{27}$, so the entire expression is also divisible by 27. Therefore, $86^n - 30^n + 21^n - 5^n$ is divisible by $8 \cdot 27 = 216$, so its remainder when divided by 216 is always 0 when $n \geq 3$.
   $\therefore$ The maximum possible remainder when dividing $86^n - 30^n + 21^n - 5^n$ by 216 is $\boxed{72}$.

2. Let $ABCD$ be a cyclic quadrilateral with circumcenter O, such that $BC = CD = 2$ and BD bisects AC. If $AC = BO$, find the area of quadrilateral $ABCD$.

   Solution:

   Let AC intersect BD at M. note that M is the midpoint of CA, and in addition $\angle CBD = \angle CDB = \angle CAB$. Thus, triangles CBM and CAB are similar, giving us that:

$$\begin{aligned} CB^2 &= CM \cdot CA \\ &= 2CM^2 \\ &= 4 \\ CM &= \sqrt{2} \\ AC &= 2\sqrt{2} \end{aligned}$$

   Thus, $OC = AC = 2\sqrt{2}$. Now, let E be the foot from O to CD. The pythagorean theorem on CEO gives us that $OE = \sqrt{7}$, and since $CO \cdot DN = OE \cdot CD$ by area, we find that $ND = \sqrt{\frac{7}{2}}$. Performing the pythagorean theorem on CND gives us that $CN = \frac{1}{\sqrt{2}}$. To finish, note that since $AM = CM$, the area of $ABCD$ is twice the area of BCD, and thus the area of $ABCD$ is:

$$2 \cdot ND \cdot CN = \boxed{\sqrt{7}}$$

3. The function $f : \mathbb{Q} \to \mathbb{Q}$ satisfies for all nonzero rationals x, y the functional equation:

$$f(x) - f\left(\frac{1}{y}\right) = \frac{xf(y) - f(1)}{y}$$

   Compute the number of functions f such that $0 < f(10) < 2020$ and $f(3)$ is an integer.

   Solution:

   Plugging in $y = 1$ into the functional equation gives us that $f(x) = xf(1)$, or $f(x) = xc$ for some constant c. Plugging this back into the function equation reveals that this solution indeed works, so we can set f as this now (note that we do not care about the value of $f(0)$ in the following calculation, although technically it could be anything).

   Note that $f(3) = 3c$ is an integer, so $c = \frac{k}{3}$ for some integer k. Since we have that $0 < f(10) < 2020$ and $f(10) = \frac{10k}{3}$, we have that $0 < k < 606$. Thus, k can be any integer from 1 to 605 inclusive, all of which give a function f which satisfies all of the given condition, so thus there are $\boxed{605}$ such functions.

Set 9

1. Compute:

$$\sum_{a=1}^{8}\sum_{b=1}^{8}\frac{a^3+b}{a+b}$$

Solution:

We have that by symmetry:

$$\begin{aligned}
\sum_{a=1}^{8}\sum_{b=1}^{8}\frac{a^3+b}{a+b} &= \frac{1}{2}\sum_{a=1}^{8}\sum_{b=1}^{8}\frac{a^3+b+b^3+a}{a+b} \\
&= \frac{1}{2}\left(\sum_{a=1}^{8}\sum_{b=1}^{8}a^2-ab+b^2+1\right) \\
&= 32+\sum_{a=1}^{8}\sum_{b=1}^{8}a^2-\frac{1}{2}\sum_{a=1}^{8}\sum_{b=1}^{8}ab \\
&= 32+8\cdot\frac{8\cdot 9\cdot 17}{6}-\frac{1}{2}(1+2+\ldots 8)^2 \\
&= 32+32\cdot 51-8\cdot 81 \\
&= \boxed{1016}
\end{aligned}$$

2. Let a_n be an infinite sequence of numbers satisfying $a_0 = 2019$ and $a_{n+1} = 10^8 \cdot a_n + 20202019$. Compute the number of digits, in base 10, of the first number in this sequence that is divisible by 99.

Solution:

In order for a_n to be divisible by 99, it must be divisible by both 9 and 11.

Using the divisibility rules for 9, we get the following equation: $12 + 16n \equiv 0 \mod 9$.

This can be rearranged to $16n \equiv -12 \mod 9$, or $7n \equiv 6 \mod 9$.

Multiplying by $7^{-1} \mod 9 = 4$, we get $28n \equiv 24 \mod 9$, or $n \equiv 6 \mod 9$. Let $n = 9m + 6$.

Using the divisibility rules for 11, we get the following equation: $6 + 2x \equiv 0 \mod 11$.

$6 + 2(9m + 6) \equiv 0 \mod 11 \implies 18m + 18 \equiv 0 \mod 11 \implies m \equiv -1 \equiv 10 \mod 11$.

Let $m = 11k + 10$. Then, we have $n = 9(11k + 10) + 6 = 99k + 96$.

Therefore, the first n for which a_n is divisible by 99 is a_{96}.

The number of digits in a_{96} is $4 + 96 * 8 = \boxed{772}$

3. Ethan Liuwu is trapped in a dungeon! He is inside a dungeon corridor, which consists of 7 tiles in a row, numbered $1, 2, \ldots, 7$ from left to right. Unfortunately, the dungeon is pitch dark and thus Ethan Liuwu moves at random. Every second, he has a $\frac{1}{2}$ chance of moving one tile to the left, a $\frac{1}{4}$ chance of moving one tile to the right, and a $\frac{1}{4}$ chance of staying on the same tile. If he reaches tile 1, he will safely leave the dungeon, never to return. However, if he reaches tile 7, he'll fall into a bottomless pit and get trapped forever. Compute the probability that Ethan Liuwu will leave the dungeon safely, given that he starts on tile 6.

Solution:

Let P_n be the probability that Ethan Liuwu exits the dungeon safely if he is on tile n. We are given that $P_1 = 1$, $P_7 = 0$, and $P_n = \frac{1}{2}P_{n-1} + \frac{1}{4}P_n + \frac{1}{4}P_{n+1}$ for all integers $1 < n < 7$.

Now, we will consider the differences $d_n = P_n - P_{n+1}$ for all integers $1 \leq n < 7$. The equation $P_n = \frac{1}{2}P_{n-1} + \frac{1}{4}P_n + \frac{1}{4}P_{n+1}$ can be rewritten as $3P_n = 2P_{n-1} + P_{n+1}$, or $P_n - P_{n+1} = 2(P_{n-1} - P_n)$. Therefore, $d_n = 2d_{n-1}$ for all $1 < n < 7$. Then, $d_2 = 2d_1$, $d_3 = 2d_2 = 4d_1$, and so on. In general, $d_n = 2^{n-1}d_1$ for all $1 \leq n < 7$.

The sum of all these differences is $d_1+d_2+\cdots+d_6 = (1+2+\cdots+2^5)d_1 = (2^6-1)d_1 = 63d_1$. However, this sum is also equal to $(P_1-P_2)+(P_2-P_3)+\cdots+(P_6-P_7) = P_1-P_7 = 1-0 = 1$. Thus, $63d_1 = 1$, so $d_1 = \frac{1}{63}$.
Finally, $d_6 = P_6 - P_7$, so $P_6 = P_7 + d_6 = 0 + 2^5 \cdot d_1 = 32d_1 = \frac{32}{63}$.

$\therefore$ Since Ethan Liuwu is on tile 6, he has a $\boxed{\dfrac{32}{63}}$ chance of exiting the dungeon alive.

Set 10

1. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. Let the internal angle bisector of $\angle BAC$ intersect the circumcircle of triangle ABC at M, and let the tangents from M to the incircle of ABC meet the circumcircle of triangle ABC at P and Q (both of which are not point M). Find the length of PQ.

   Solution:

   The pith of this problem is the following lemma:

   LEMMA: Let I be the incenter of ABC. Then if $AB + AC = 2BC$, then $AI = IM$.

   For a proof of this lemma, please refer to HMMT February 2013 Team 6 (this lemma was likely known before then, but this is a good place to look for a variety of solutions of this lemma).

   Given this, let O be the circumcenter of ABC. Since $4 + 6 = 2 \cdot 5$, we have that $AI = IM$ and thus OI is perpendicular to AM. This gives us that A and M are reflections with relation to OI, and the incircle is also symmetric with relation to OI.

   Now, given this we find that in fact $PQ = BC$ by symmetry, and thus our answer is $PQ = BC = \boxed{5}$

2. Find the smallest integer $x > 11$ such that there exists an integral n such that:

$$S = \frac{\log_{x-10}(n-10)}{\log_x n}$$

   is a rational number greater than 1.

   CLARIFICATION: both the numerator and the denominator should be rational (we do not know the answer otherwise). This clarification was not given in-contest, but this does not seem to have affected to the in-contest results.

   Solution: First of all, given that both the numerator and the denominator both have to be rational, they must both be integral.

   Now, say $x = 12$ gives a solution in n. Then $n - 10$ must be a power of two, n must be a power of 12, and n must be larger than 12. Thus, $\log_{12} n$ and $\log_2 n - 10$ must both be at least 2. But this means that both n and $n - 10$ must be divisible by 4, a contradiction given that their difference is 10.

   Now, for $x = 13$ $n = 2197$ works. Thus our answer is $\boxed{13}$

3. Find the probability that for any positive integer n, we have that 61 divides:

$$20^n - 210^n + 70^n - 125^n + 245^n$$

   Solution:

   We do everything mod 61. Note that the given condition is equivalent to:

$$\begin{aligned} 0 &\equiv 20^n - 210^n + 70^n - 125^n + 245^n \pmod 61 \\ &\equiv 81^n - 27^n + 9^n - 3^n + 1^n \pmod 61 \\ &\equiv \frac{3^{5n}+1}{3^n+1} \pmod 61 \end{aligned}$$

   Note that $3^5 = 243 \equiv -1 \pmod 61$, so $243^n + 1$ is divisible by 61 for all odd n. However, if n is divisible by 5, then we can evaluate that $81^n - 27^n + 9^n - 3^n + 1^n \equiv 1 \pmod 61$. Thus, the only positive integral n such that 61 divides $20^n - 210^n + 70^n - 125^n + 245^n$ are those that are odd but not divisible by 5. Thus, all such integers are equivalent to 1, 3, 7, or 9 $\pmod 10$, giving us that the probability that any chosen positive n satisfies the condition is $\boxed{\frac{2}{5}}$