UTMC Junior Team Round Solutions

UTMC Committee

March 2020

1. One day, Charles played around with the letters of his favourite word, $ROSEBUD$. In how many ways could he arrange the letters of $ROSEBUD$ such that:

   - the letter R appears somewhere before the letter B, and
   - no two vowels are adjacent to each other?

   Solution:

   First, by Stars and Bars, the number of ways to choose 3 positions for the vowels such that none of them are adjacent to each other is $\binom{5}{3} = 10$. We also have to permute the vowels, and there are $3! = 6$ ways to do this. Then, we have to arrange the consonants, and there are $4! = 24$ ways in total to do so. However, by symmetry, only half of these ways have R appearing before B, and the other half have R appearing after B. Thus, there are $\frac{24}{2} = 12$ valid ways to arrange the consonants after the vowels are arranged.
   Finally, by the Fundamental Counting Principle, we multiply to get $10 \cdot 6 \cdot 12 = 720$.
   $\therefore$ The number of valid ways to arrange the letters of $ROSEBUD$ is $\boxed{720}$.

2. David is playing a casino game. Each round, he has a $\frac{1}{3}$ chance of winning a jackpot revenue of 2000 tickets. If he does not win this jackpot, he wins a revenue of either 5 tickets or 15 tickets, with an equal chance of winning each prize. If David plays exactly 3 times, and the number of tickets he owns does not change between games, compute the probability that his total revenue is at least 2020 tickets.

   Solution:

   First, if David wins 2 or 3 jackpots, he earns more than $2 \cdot 2000 = 4000 > 2020$ tickets. He has a $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$ chance of winning 3 jackpots, and he has a $\left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \cdot \binom{3}{2} = \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} \cdot 3 = \frac{2}{9}$ chance of winning exactly 2 jackpots. Additionally, if David does not win any jackpots, he earns at most $3 \cdot 15 = 45 < 2020$ tickets.
   Now, suppose that David wins exactly one jackpot. This has a $\frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 \cdot \binom{3}{1} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 \cdot 3 = \frac{4}{9}$ chance of happening. However, if he wins two prizes of 5 tickets, he earns $2000 + 5 + 5 = 2010$ tickets, which is not enough. Thus, he needs to win at least one prize of 15 tickets. By complementary counting, this has a $1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$ chance of happening, since he has a $\frac{1}{2}$ chance of winning a prize of 5 tickets whenever he does not win a jackpot. Overall, the probability that David wins exactly one jackpot and earns at least 2020 tickets is $\frac{4}{9} \cdot \frac{3}{4} = \frac{1}{3}$.

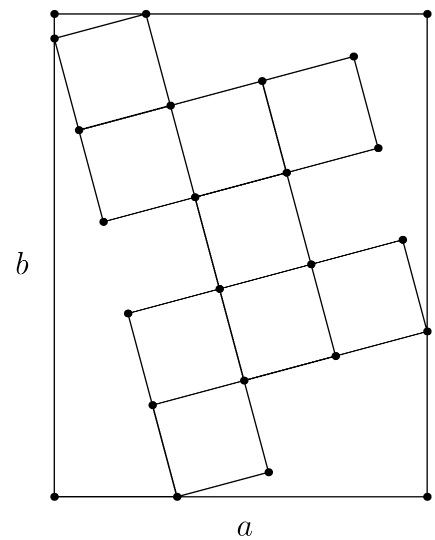
   $\therefore$ The total probability that David wins at least 2020 tickets is $\frac{1}{27} + \frac{2}{9} + \frac{1}{3} = \boxed{\frac{16}{27}}$.

3. Find all nonnegative integers n such that removing the last digit of $(3n+2)^2$ results in the number n^2.

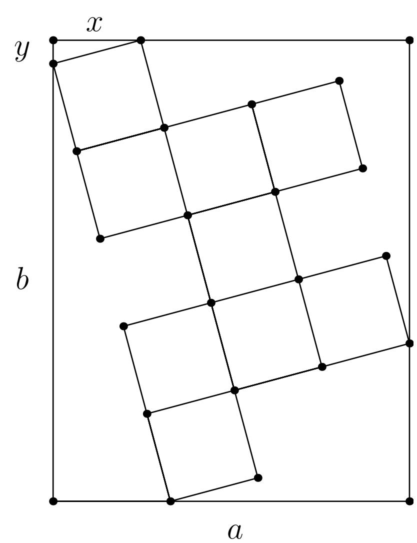
Solution:

If the last digit is k, we want $(3n+2)^2 = 10n^2 + k$. Solving, we have $k = -n^2 + 12n + 4$. Notice that $0 \le k \le 9$ which forces $0 \le n \le 12$. Checking all of these, only $\boxed{n = 0, 12}$ work.

4. The figure below consists of 9 congruent squares inscribed within a larger rectangle. If $a = 126$ and $b = 159$, compute the area of one of the 9 congruent squares.



Solution:



Label x and y as in the diagram above.

Then, we have $a = 3(x + y) + y = 3x + 4y = 126$ and $b = 5x + y = 159$.

Solving the system of equations, we get $x = 30, y = 9$.

The area of a square is $s^2 = x^2 + y^2 = \boxed{981}$

5. We have 3 prime numbers p, q, and r such that $p+q+r=50$ and $pq+qr+rp$ is the maximum possible for all such triplets of primes. Find pqr.

   Solution:

   Note that since $p+q+r=50$, at least one of p,q,r must be even. Since there exists only one even prime, we will WLOG set $p=2$. Now, we have that $q+r=50$, and we wish to maximize $qr+2q+2r=qr+100$. Note that by AM-GM, this sum is maximized when $|q-r|$ is as small as possible, which occurs with the pair of primes $(19,29)$. Thus, we have that our answer is:

$$\begin{aligned} pqr &= 2 \cdot 19 \cdot 29 \\ &= \boxed{1102} \end{aligned}$$

6. A tower with length and width of 2 and height of 100 is constructed with 1 by 1 by 1 blocks. Then, a colour of either red or blue assigned at random, with each block being coloured 1 colour. What is the expected number of adjacent pairs of blocks with opposite colours?

Solution:

We use linearity of expectation. For any adjacent pair of blocks, the probability that they are of different colors is $\frac{1}{2}$. Now, the number of adjacent pairs per "level" is 4, and the number of "vertical" adjacent pairs is "$4 \cdot 99$. This gives us that the number of adjacent pairs in the tower is:

$$\begin{aligned} 4 \cdot 99 + 100 \cdot 4 &= 4 \cdot 199 \\ &= 796 \end{aligned}$$

Thus, by linearity of expectation we have that the expected number of adjacent pairs of blocks with opposite colors is:

$$796 \cdot \frac{1}{2} = \boxed{398}$$

7. The sequence (x_n) is defined by $x_0 = 1$ and $x_{n+1} = 2022^n + x_n$ for all $n \geq 0$. Compute the last two digits of x_{2020}.

Solution:

Define $a_n = \frac{x_n}{2022^n}$. Then, we have $2022a_{n+1} = a_n + 1$ for all $n \geq 0$.

This rearranges to $2022\left(a_{n+1} - \frac{1}{2021}\right) = a_n - \frac{1}{2021}$, so we define $b_n = a_n - \frac{1}{2021}$.

Therefore, $2022b_{n+1} = b_n$, so $b_n = b_0 \cdot 2022^{-n}$ and $a_n = b_0 \cdot 2022^{-n} + \frac{1}{2021}$

$x_n = 2022^n\left(b_0 \cdot 2022^{-n} + \frac{1}{2021}\right) = b_0 + \frac{2022^n}{2021}$

Since $x_0 = b_0 + \frac{1}{2021} = 1$, we have $b_0 = \frac{2020}{2021}$ and $x_n = \frac{2022^n + 2020}{2021}$.

Now, we wish to find the modular inverse of 2021 mod 100. We note that this is equivalent to finding the modular inverse of $3 \cdot 7 = 21$ mod 100

Since $3 \cdot 67 = 201 \equiv 1 \bmod 100$ and $7 \cdot 43 = 301 \equiv 1 \bmod 100$, we see that $21^{-1} \bmod 100 = 67 \cdot 43 = 2881 \equiv 81 \bmod 100$.

$x_{2020} \equiv 81\left(2022^{2020} + 2020\right) \equiv 81\left(22^{2020} + 20\right) \bmod 100$.

Note that $22^{2020} \equiv 2^{2020} \cdot 11^{2020} \equiv 2^{2020} \cdot 11^{2020 \bmod \phi(100)} \equiv 2^{2020} \cdot 11^{20} \bmod 100$.

Since $2^{2020} \equiv 0 \bmod 4$ and $2^{2020} \equiv 2^{2020 \bmod \phi(25)} \equiv 1 \bmod 25$, $2^{2020} \equiv 76 \bmod 100$.

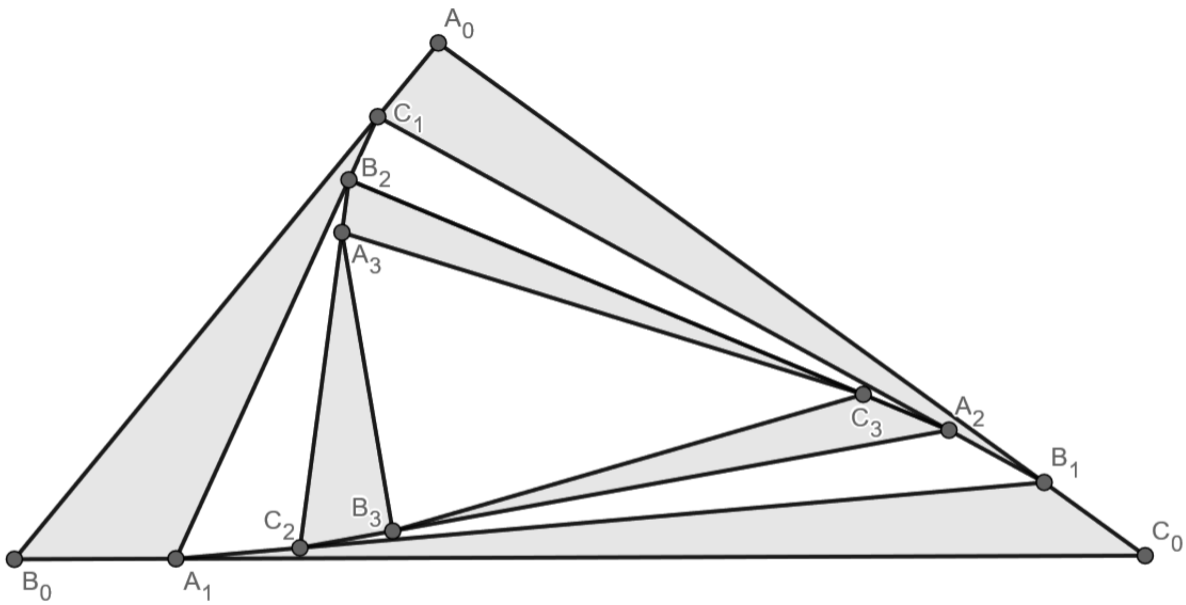
Since $20 \equiv 0 \bmod 10$, $11^{20} = (10+1)^{20} = \sum_{i=0}^{20}\left(\binom{20}{i}10^i\right) \equiv 1 \bmod 100$.

Therefore, $22^{2020} = 2^{2020} \cdot 11^{2020} \equiv 76 \bmod 100$.

$x_{2020} \equiv 81(76 + 20) \equiv \boxed{76} \bmod 100$

8. Let triangle $\Delta A_0B_0C_0$ have an area of 400. For all integers $k > 0$, construct $\Delta A_kB_kC_k$ recursively as follows: let A_k, B_k, and C_k be points on sides $B_{k-1}C_{k-1}$, $C_{k-1}A_{k-1}$, and $A_{k-1}B_{k-1}$, respectively, such that:
$$\frac{A_kB_{k-1}}{A_kC_{k-1}} = \frac{B_kC_{k-1}}{B_kA_{k-1}} = \frac{C_kA_{k-1}}{C_kB_{k-1}} = \frac{1}{6}$$
Compute the total area covered by all the points that are inside an odd number of these triangles.

Solution:



As shown in the diagram, we can find the shaded area by subtracting the area of $\Delta A_1B_1C_1$ from the area of $\Delta A_0B_0C_0$, then adding the area of $\Delta A_2B_2C_2$, then subtracting the area of $\Delta A_3B_3C_3$, and so on. Thus, if we know the ratio between the areas of two consecutive triangles, then we can express the shaded area as an infinite geometric series with a negative common ratio.

For any positive integer k, we are given that:
$$\frac{A_kB_{k-1}}{A_kC_{k-1}} = \frac{1}{6}$$
We can use this to get:
$$\frac{B_{k-1}C_{k-1}}{A_kC_{k-1}} = \frac{A_kC_{k-1} + A_kB_{k-1}}{A_kC_{k-1}} = 1 + \frac{1}{6} = \frac{7}{6}$$
Thus, $A_kC_{k-1} = \frac{6}{7}B_{k-1}C_{k-1}$, and $A_kB_{k-1} = B_{k-1}C_{k-1} - A_kC_{k-1} = \frac{1}{7}B_{k-1}C_{k-1}$. Similarly, $B_kA_{k-1} = \frac{6}{7}C_{k-1}A_{k-1}$, $B_kC_{k-1} = \frac{1}{7}C_{k-1}A_{k-1}$, $C_kB_{k-1} = \frac{6}{7}A_{k-1}B_{k-1}$, and $C_kA_{k-1} = \frac{1}{7}A_{k-1}B_{k-1}$.

Now, let $|\Delta XYZ|$ denote the area of an arbitrary triangle XYZ. Then, we can use the well-known formula $|\Delta XYZ| = \frac{1}{2}XY \cdot XZ \cdot \sin\angle YXZ$ to get:
$$\frac{|\Delta A_{k-1}B_kC_k|}{|\Delta A_{k-1}B_{k-1}C_{k-1}|} = \frac{\frac{1}{2}A_{k-1}B_k \cdot A_{k-1}C_k \cdot \sin\angle B_{k-1}A_{k-1}C_{k-1}}{\frac{1}{2}A_{k-1}C_{k-1} \cdot A_{k-1}B_{k-1} \cdot \sin\angle B_{k-1}A_{k-1}C_{k-1}} = \frac{6}{7} \cdot \frac{1}{7} = \frac{6}{49}$$
Thus, $|\Delta A_{k-1}B_kC_k| = \frac{6}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|$.

Similarly, $|\Delta B_{k-1}C_kA_k| = |\Delta C_{k-1}A_kB_k| = \frac{6}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|$. Then, we get:

$$\begin{aligned}|\Delta A_kB_kC_k| &= |\Delta A_{k-1}B_{k-1}C_{k-1}| - |\Delta A_{k-1}B_kC_k| - |\Delta B_{k-1}C_kA_k| - |\Delta C_{k-1}A_kB_k| \\ &= \left(1 - 3\cdot\frac{6}{49}\right)|\Delta A_{k-1}B_{k-1}C_{k-1}| \\ &= \frac{31}{49}|\Delta A_{k-1}B_{k-1}C_{k-1}|\end{aligned}$$

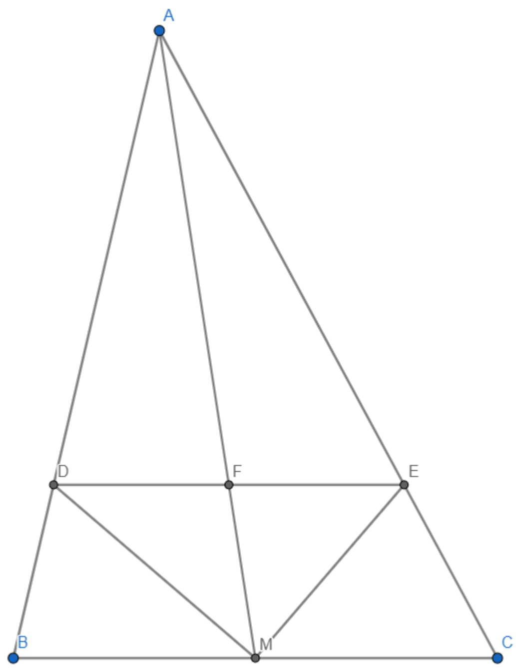
Therefore, in our sequence of triangles, the ratio between two consecutive areas is $\frac{31}{49}$. Finally, since the area of our initial triangle $A_0B_0C_0$ is 400, our infinite geometric series is:

$$\begin{aligned}400\left(1 - \frac{31}{49} + \left(\frac{31}{49}\right)^2 - \left(\frac{31}{49}\right)^3 + \cdots\right) &= 400\left(1 + \left(-\frac{31}{49}\right) + \left(-\frac{31}{49}\right)^2 + \cdots\right) \\ &= \frac{400}{1-(-\frac{31}{49})} \\ &= \frac{400}{\frac{80}{49}} \\ &= 245\end{aligned}$$

$\therefore$ A total area of $\boxed{245}$ square units is contained inside an odd number of triangles.

9. Let ABC be a triangle with $AB = 7$, $BC = 4$, and $CA = 9$. Let M be the midpoint of BC. Let the internal angle bisector of $\angle AMB$ intersect AB at D, and let the internal angle bisector of $\angle AMC$ intersect AC at E. Finally, let F be the midpoint of DE. Find AF.

Solution:



We first calculate AM. Set $AM = x$, and note that $BM = CM = 2$. Then, note that by Stewart's Theorem we have:

$$\begin{aligned} 4x^2 + 2 \cdot 4 \cdot 2 &= 7^2 \cdot 2 + 9^2 \cdot 2 \\ 4x^2 + 16 &= 98 + 162 \\ 4x^2 &= 98 + 162 - 16 \\ &= 244 \\ x &= \sqrt{61} \end{aligned}$$

Now, the next part of this problem relies on the following synthetic observations. Note that by the Angle Bisector Theorem, we have that:

$$\begin{aligned} \frac{AD}{DB} &= \frac{AM}{MB} \\ &= \frac{AM}{MC} \\ &= \frac{AE}{EC} \end{aligned}$$

Thus, we have that DE and BC are parallel, and in fact triangles ADE and ABC are homothetic with relation to A. This gives us that in fact F also lies on AM, and we have the relation:

$$\begin{aligned}\frac{AF}{FM} &= \frac{AD}{DB} \\ &= \frac{AM}{MB}\end{aligned}$$

Finally, we compute:

$$\begin{aligned}\frac{AF}{AM} &= \frac{AM}{AM+MB} \\ AF &= \frac{AM^2}{AM+MB} \\ &= \boxed{\frac{61}{2+\sqrt{61}}}\end{aligned}$$

NOTE: originally, in the original copy of this question we had swapped BC and CA: this would have led to the answer of $\frac{49}{32}$, which was the answer used during scoring at the time. Unfortunately, this text correction was not displayed on the in-contest problem sheet, and thus an incorrect answer was used for scoring.

10. We place a knight in a 3x3 grid such that it has at least one valid move (a knight, as defined in chess, can move two steps horizontally in either direction and one step vertically in either direction, or two steps vertically and one step horizontally). How many ways can we make 16 moves with it such that it ends up in the same square it starts on?

Solution: The 8 possible squares the knight can be on form a cycle, so the answer is just $16C0 + 16C4 + 16C8 + 16C12 + 16C16$, as the number of moves counterclockwise in the cycle of length 8 must be a multiple of 4. This sum can be evaluated using roots of unity filter as the expression $\frac{2^{16}+(1+i)^{16}+(1-i)^{16}}{4}$ or long multiplication to be $\boxed{16512}$.