UTMC Junior Individual Round Solutions

UTMC Committee

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1. What is the maximum value of $a^{b^{c^d}}$, given that a, b, c, d are all distinct integers chosen from the set $(0, 1, 2, 9)$?

   Solution:

   First, we obviously want $a > 1$, so $a = 2$ or 9. It is easy to see that in each case, the maximum possible value of the exponent b^{c^d} is $9^{1^0} = 9$ or $2^{1^0} = 2$, respectively. This gives us maximum values of $2^9 = 512$ and $9^2 = 81$, respectively. The larger of these two values is $\boxed{512}$.

2. Sonic the Hedgehog runs at a constant speed of 5 km/hr. He needs to run 40 km, and along the way he must use 3 boosts that double his speed for the next 5 km (if he uses these speed boosts after he has already run 35 km, then they will just terminate when he runs his total of 40 km). These speed boosts can stack on top of each other: if two are used at the same time, they will increase his speed by a factor of 4 instead of 2. What is the difference between the fastest and slowest times he can take to complete the race?

   Solution:

   First, we notice that using the boosts in the last second will make them essentially useless. The best way to use the boosts is to stagger them, as the boosts are distance and not time based, so the slower you go, the more you get from the boosts. Thus, the difference in the times is whether we double our speed or not for the first 15 kilometers, or $\frac{15}{5} * \frac{1}{2} = \boxed{\frac{3}{2}}$.

3. Define the operation $\diamond$ such that:

$$a \diamond b = ab - 9a + 9$$

Compute:

$$(\ldots((1 \diamond 2) \diamond 3) \ldots \diamond 9) \diamond 10$$

Solution:

The key idea of this problem is that $a \diamond 9 = 9a - 9a + 9 = 9$. This reduces the given expression to

$$9 \diamond 10 = 9 \cdot 10 - 9 \cdot 9 + 9 = \boxed{18}$$

4. Let d be a randomly selected positive divisor of 6^6. Compute the expected number of positive divisors of d.

Solution:

Since $6^6 = 2^6 \cdot 3^6$, the prime factorization of d must be in the form $2^m \cdot 3^n$, where m and n are nonnegative integers such that $m \le 6$ and $n \le 6$. Then, d has $(m+1)(n+1)$ positive divisors. Since there are $(6+1)(6+1) = 49$ possible values of d, the expected number of positive divisors of d is:

$$\begin{aligned}\frac{1}{49}\sum_{m=0}^{6}\sum_{n=0}^{6}(m+1)(n+1) &= \frac{1}{49}\left(\sum_{m=0}^{6}(m+1)\right)\sum_{n=0}^{6}(n+1)\\ &= \frac{1}{49}(1+2+\cdots+7)(1+2+\cdots+7)\\ &= \frac{1}{49}\cdot\frac{7\cdot 8}{2}\cdot\frac{7\cdot 8}{2}\\ &= \frac{8}{2}\cdot\frac{8}{2}\\ &= 16\end{aligned}$$

$\therefore$ The expected number of positive divisors of d is $\boxed{16}$.

5. Pete Alonso, a baseball player, is good at hitting home runs. Over the long term, a certain positive proportion of his games have home runs. Right after he plays a game in which he hits a home run, his confidence increases, and he becomes 2 times as likely, compared to average, to hit a home run in the next game. However, if he plays a game without hitting a home run, he is only $\frac{1}{2}$ as likely as average to hit a home run in the next game. On average, what proportion of his games contain home runs?

Solution:

Suppose that the answer is x. Then, the probability that Alonso hits a home run in any arbitrary game is x.
On the other hand, we can compute this same probability using casework on the previous game. There is a probability of x that the previous game contains a home run from Alonso, and then he has a $2x$ chance of hitting a home run in this game. There is also a probability of $1 - x$ that he did not hit a home run on the previous game, and then he has a $\frac{1}{2}x$ chance of hitting a home run in this current game. In total, the probability that he hits a home run is $x \cdot 2x + (1 - x) \cdot \frac{1}{2}x$. Then,

$$x = x \cdot 2x + (1 - x) \cdot \frac{1}{2}x$$

$$x = 2x^2 + (\frac{1}{2}x - \frac{1}{2}x^2)$$

$$\frac{3}{2}x^2 - \frac{1}{2}x = 0$$

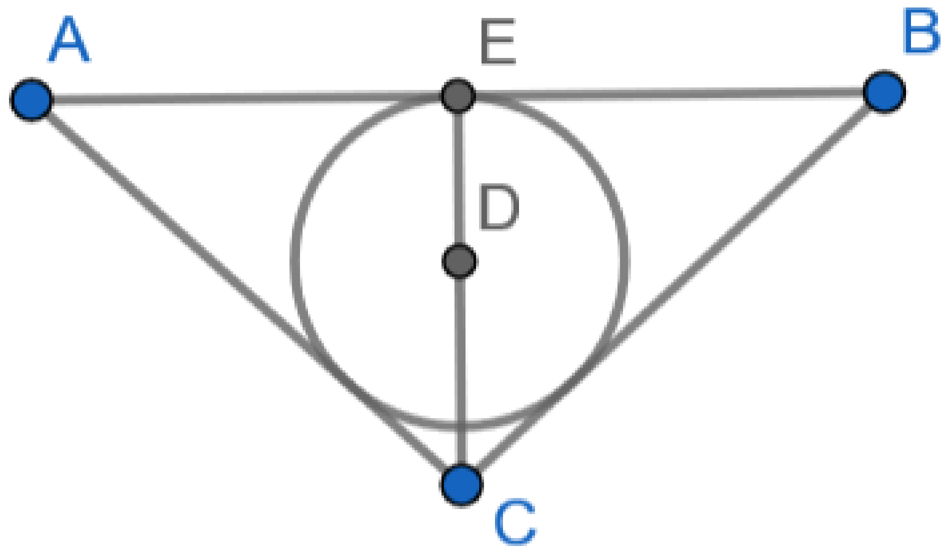
$$\frac{1}{2}x(3x - 1) = 0$$

From factoring, we get that $x = 0$ or $x = \frac{1}{3}$. We are given that x is positive, so the answer is $\boxed{\frac{1}{3}}$.

6. Ethan is out enjoying some ice-cold ice-cream. The ice cream consists of one ice cream cone, with one spherical ice cream scoop resting within it. The cone has a height of 32 and a base radius of 60. Looking horizontally at the rim of the cone, Ethan noticed that the scoop could barely be seen. Given that the scoop never melts (and thus preserves its spherical shape), what is the volume of the ice cream scoop?

   Solution:

   Clearly, since the ice-cream scoop can barely be seen, it is tangent to the base of the ice-cream cone. Then, consider the following vertical cross-section of the sphere and the cone, where point C is the cone's vertex, point D is the sphere's centre, and point E is the tangency point.



   Clearly, the radius of the circle is the radius of the sphere, and the circle is the incircle of triangle ABC. Then, we should find the inradius by dividing the area by the semiperimeter.
   We are given that $EC = 32$ and $EA = EB = 60$. Then, the area is simply $\frac{bh}{2} = \frac{AB \cdot EC}{2} = \frac{120 \cdot 32}{2} = 1920$. Moreover, by the Pythagorean Theorem, $AC = BC = \sqrt{EB^2 + EC^2} = \sqrt{60^2 + 32^2} = 68$. Then, the semiperimeter is $\frac{AB+BC+CA}{2} = \frac{120+68+68}{2} = 128$. Thus, the inradius of triangle ABC is $\frac{1920}{128} = 15$.
   Finally, the volume of the ice-cream scoop is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(15)^3 = \boxed{4500\pi}$.

7. For non-zero real numbers m and n, define the operation ★ as $m \bigstar n = \frac{1}{m} + \frac{1}{n} - \frac{1}{mn}$. Compute the value of

$$4\bigstar(5\bigstar(\dots(2018\bigstar 2019)\dots))$$

Solution:

We work backwards starting from the inside bracket and moving out. For the first term, we find $\frac{1}{2018} + \frac{1}{2019} - \frac{1}{2018*2019} = \frac{2}{2019}$. The second term is $\frac{1}{2017} + \frac{2019}{2} - \frac{2019}{2*2017} = 1009$. Now that we've established out base cases, we can show that for all n, $\frac{1}{2020-2n} + \frac{1}{\frac{2020-n}{n}} - \frac{1}{(2020-2n)\frac{2020-n}{n}} = \frac{n+1}{2020-n}$ and that $\frac{1}{2020-2n-1} + \frac{1}{\frac{n+1}{2020-n}} - \frac{1}{(2020-2n-1)\frac{n+1}{2020-n}} = \frac{2020-n-1}{n+1}$. Thus, it's clear by induction that for all even integers $2020-2n$ on the left side, the final result would be $\frac{n+1}{2020-n}$. Thus, for $2020-2n=4$, we have that $n=1008$ so our final answer is $\boxed{\frac{1009}{1012}}$

8. Given that there exists four distinct primes for which the sum of any three is a prime number, find the minimum possible value of the smallest prime out of the four.

   Solution:

   If the smallest prime is 3, then the other three primes must be either 1 or 2 modulo 3. We find that for any combination of other primes, we can always find three of these primes that sum to a multiple of 3, while this sum is obviously > 3 so it cannot be prime. Thus, the smallest prime must be 5. The construction for this is the primes $5, 7, 17, 19$.

9. The company MMM (Make More Money) made a whopping profit of \$20 last year, and the N owners are splitting the money using the following process. The youngest owner proposes a plan where each owner receives a positive integer number of dollars, such that the youngest owner maximizes their own profit while still satisfying the other owners. After the split, each owner votes. (Assume that all owners have different ages.) If at most 1 owner votes against the plan, it passes; otherwise, the youngest owner gets fired and receives \$0, and the process repeats for the remaining $N-1$ owners. Each owner tries to maximize the amount of money they receive; if there are multiple possible moves that do this, they try to fire as many owners as possible. Assume that the owners do not collude, but that they act optimally otherwise. Given that $N=4$, compute the amount of money that the youngest owner receives.

   Solution:

   First, if $N=2$, then the youngest owner maximizes their amount of money by giving the oldest owner only \$1 and keeping the rest of the money. Even if the oldest owner votes against this plan, the youngest owner does not get fired.
   Then, if $N=3$, the youngest owner can give the oldest owner only \$2, then the oldest owner will not vote against the plan (otherwise, they will only earn \$1.) The youngest owner can then receive \$17, leaving only \$1 for the middle owner.
   Similarly, if $N=4$, the youngest owner can give \$3 and \$2 to the two oldest owners, and then they will not vote against this plan. The second-youngest owner must receive \$1, then the youngest owner happily earns \$14.
   $\therefore$ When $N=4$, the youngest owner earns $\boxed{\$14}$.

10. Let ABC be a triangle satisfying $AB = 3$, $AC = 5$, $BC = 7$. Let A' be the reflection of A across BC, and let the tangents to the circumcircle of triangle ABC from points B and C intersect at a point K. Find $A'K$.

Solution:

The key idea behind this problem is to note that if we let $\cos \angle BAC = x$, then we have by cosine law that:

$$\begin{aligned} 7^2 &= 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot x \\ 49 &= 34 - 30x \\ x &= -\frac{1}{2} \\ \angle BAC &= 120^\circ \end{aligned}$$

This gives us that $\angle BA'C = 120^\circ$, and $\angle KBC = \angle KCB = 180^\circ - \angle BAC = 60^\circ$. This in fact indicates that KBC is equilateral, and now $\angle BKC = 60^\circ = 180^\circ - \angle BA'C$ gives us that in fact $BKCA'$ is cyclic.

Now, we apply Ptolemy to $BKCA'$ (or the fabled van Schooten theorem) to find that:

$$\begin{aligned} BK \cdot A'C + CK \cdot A'B &= BC \cdot A'K \\ A'K &= A'C + A'B \\ &= AC + AB \\ &= 5 + 3 \\ &= \boxed{8} \end{aligned}$$

11. Michael is playing a game with his 4 friends to split a single cake. To begin, each player flips one coin. Then, all people who flip "Heads" share the cake equally. If everyone flips "Tails", the cake is thrown out and nobody gets to eat it. What is the expected amount of cake that Michael will get?

Solution:

By symmetry, each of the 5 players will get an equal expected amount of cake. Thus, we simply need to find the expected amount of cake that is given out, then divide by 5.
No cake is given out if everyone flips "Tails", and the entire cake is given out otherwise. There is only a $\frac{1}{2^5} = \frac{1}{32}$ chance that everyone flips "Tails", so the expected amount of cake is $1 - \frac{1}{32} = \frac{31}{32}$.
$\therefore$ The expected amount of cake received by Michael is $\frac{31}{32} \cdot \frac{1}{5} = \boxed{\frac{31}{160}}$.

12. Let x be a real number such that $\log_4(x), \log_8(4x),$ and $2020 + \log_{32}(x)$ are three consecutive terms in a geometric sequence (in that order). The product of all possible values of x can be written in the form 2^n for some real number n. Compute n.

Solution:

To be consecutive terms in a geometric sequence, $\log_4(x), \log_8(4x),$ and $2020 + \log_{32}(x)$ must satisfy the relation

$$\log_4(x)\left(2020 + \log_{32}(x)\right) = \left(\log_8(4x)\right)^2$$

Converting all logs to base 2, we get

$$\frac{1}{2}\log_2(x)\left(2020 + \frac{1}{5}\log_2(x)\right) = \left(\frac{1}{3}\log_2(4x)\right)^2$$

Setting $y = \log_2(x)$, this simplifies to

$$\frac{y}{2}\left(2020 + \frac{y}{5}\right) = \left(\frac{y}{3} + \frac{2}{3}\right)^2$$

Expand and rearrange to get

$$\frac{y^2}{10} + 1010y = \frac{y^2}{9} + \frac{4y}{9} + \frac{4}{9}$$
$$y^2 - 90860y + 40 = 0$$

Since the discriminant of this is clearly larger than 0, there are 2 real solutions to the quadratic. Let these solutions be y_1 and y_2.

Also since $y = \log_2(x)$, we have $x = 2^y$.

Then, the product of the solutions is $x_1 x_2 = 2^{y_1} \cdot 2^{y_2} = 2^{y_1+y_2}$.

By Vieta's this is 2^{90860} and our answer is $\boxed{90860}$