UTMC Senior Team Round

1. (4) David is playing a casino game. Each round, he has a $\frac{1}{3}$ chance of winning a jackpot revenue of 2000 tickets. If he does not win this jackpot, he wins a revenue of either 5 tickets or 15 tickets, with an equal chance of winning each prize. If David plays exactly 3 times, and the number of tickets he owns does not change between games, compute the probability that his total revenue is at least 2020 tickets.

2. (6) We have 3 distinct prime numbers p, q, and r such that $p + q + r = 50$ and $pq + qr + rp$ is the maximum possible for all such triplets of primes. Find pqr.

3. (6) A tower with length and width of 2 and height of 100 is constructed with 1 by 1 by 1 blocks. Then, a colour of either red or blue assigned at random, with each block being colored 1 color. What is the expected number of adjacent pairs of blocks with opposite colours?

4. (7) The sequence (x_n) is defined by $x_0 = 1$ and $x_{n+1} = 2022^n + x_n$ for all $n \geq 0$. Compute the last two digits of x_{2020}.

5. (7) Let triangle $\Delta A_0B_0C_0$ have an area of 400. For all integers $k > 0$, construct $\Delta A_kB_kC_k$ recursively as follows: let A_k, B_k, and C_k be points on sides $B_{k-1}C_{k-1}$, $C_{k-1}A_{k-1}$, and $A_{k-1}B_{k-1}$, respectively, such that:
$$\frac{A_kB_{k-1}}{A_kC_{k-1}} = \frac{B_kC_{k-1}}{B_kA_{k-1}} = \frac{C_kA_{k-1}}{C_kB_{k-1}} = \frac{1}{6}$$
Compute the total area covered by all the points that are inside an odd number of these triangles.

6. (8) For all integers $1 \leq n \leq 18$, define $f(n)$ to be the smallest positive integer k such that nk is congruent to either 1 or 18 modulo 19. Compute:
$$\sum_{n=1}^{18} f(n)$$

7. (8) Let ABC be a triangle with $AB = 7$, $BC = 4$, and $CA = 9$. Let M be the midpoint of BC. Let the internal angle bisector of $\angle AMB$ intersect AB at D, and let the internal angle bisector of $\angle AMC$ intersect AC at E. Finally, let F be the midpoint of DE. Find AF.

8. (8) We place a knight in a 3x3 grid such that it has at least one valid move (a knight, as defined in chess, can move two steps horizontally in either direction and one step vertically in either direction, or two steps vertically and one step horizontally). How many ways can we make 16 moves with it such that it ends up in the same square it starts on?

9. (9) Michael repeatedly generates random integers between 1 and 20, inclusive, and writes them in order until the number 20 is generated. Compute the probability that he can then delete some written numbers (possibly none) until there are 10 distinct odd numbers followed by 10 distinct even numbers.

10. (10) Let ABC be a triangle, and let E and F be the feet of the altitudes from B to AC and C to AB respectively. Let D be the foot of the altitude from A to BC. Say line FD intersects the circumcircle of triangle ADB at $P \neq D$, and say ED intersects the circumcircle of triangle ADC at $Q \neq D$. Let X be the incenter of triangle DPE and Y be the incenter of triangle DQF. If $AB = 13$, $AC = 15$, and $BC = 14$, compute XY.