UTMC Junior Individual Round

1. (4) What is the maximum value of $a^{b^{c^d}}$, given that a, b, c, d are all distinct integers chosen from the set $(0, 1, 2, 9)$?

2. (5) Sonic the Hedgehog runs at a constant speed of 5 km/hr. He needs to run 40 km, and along the way he must use 3 boosts that double his speed for the next 5 km (if he uses these speed boosts after he has already run 35 km, then they will just terminate when he runs his total of 40 km). These speed boosts can stack on top of each other: if two are used at the same time, they will increase his speed by a factor of 4 instead of 2. What is the difference between the fastest and slowest times he can take to complete the race?

3. (6) Define the operation $\diamond$ such that:

$$a \diamond b = ab - 9a + 9$$

   Compute:

$$(\ldots((1 \diamond 2) \diamond 3) \ldots \diamond 9) \diamond 10$$

4. (7) Let d be a randomly selected positive divisor of 6^6. Compute the expected number of positive divisors of d.

5. (7) Pete Alonso, a baseball player, is good at hitting home runs. Over the long term, a certain positive proportion of his games have home runs. Right after he plays a game in which he hits a home run, his confidence increases, and he becomes 2 times as likely, compared to average, to hit a home run in the next game. However, if he plays a game without hitting a home run, he is only $\frac{1}{2}$ as likely as average to hit a home run in the next game. On average, what proportion of his games contain home runs?

6. (7) Ethan is out enjoying some ice-cold ice-cream. The ice cream consists of one ice cream cone, with one spherical ice cream scoop resting within it. The cone has a height of 32 and a base radius of 60. Looking horizontally at the rim of the cone, Ethan noticed that the scoop could barely be seen. Given that the scoop never melts (and thus preserves its spherical shape), what is the volume of the ice cream scoop?

7. (7) For non-zero real numbers m and n, define the operation $\bigstar$ as $m \bigstar n = \frac{1}{m} + \frac{1}{n} - \frac{1}{mn}$. Compute the value of

$$4\bigstar(5\bigstar(\ldots(2018\bigstar 2019)\ldots))$$

8. (8) Given that there exists four primes for which the sum of any three is a prime number, find the minimum possible value of the smallest prime out of the four.

9. (8) The company MMM (Make More Money) made a whopping profit of \$20 last year, and the N owners are splitting the money using the following process. The youngest owner proposes a plan where each owner receives a positive integer number of dollars, such that the youngest owner maximizes their own profit while still satisfying the other owners. After the split, each owner votes. (Assume that all owners have different ages.) If at most 1 owner votes against the plan, it passes; otherwise, the youngest owner gets fired and receives \$0, and the process repeats for the remaining $N-1$ owners. Each owner tries to maximize the amount of money they receive; if there are multiple possible moves that do this, they try to fire as many owners as possible. Assume that the owners do not collude, but that they act optimally otherwise. Given that $N=4$, compute the amount of money that the youngest owner receives.

10. (9) Let ABC be a triangle satisfying $AB=3$, $AC=5$, $BC=7$. Let A' be the reflection of A across BC, and let the tangents to the circumcircle of triangle ABC from points B and C intersect at a point K. Find $A'K$.

11. (10) Michael is playing a game with his 4 friends to split a single cake. To begin, each player flips one coin. Then, all people who flip "Heads" share the cake equally. If everyone flips "Tails", the cake is thrown out and nobody gets to eat it. What is the expected amount of cake that Michael will get?

12. (10) Let x be a real number such that $\log_4(x), \log_8(4x)$, and $2020+\log_{32}(x)$ are three consecutive terms in a geometric sequence (in that order). The product of all possible values of x can be written in the form 2^n for some real number n. Compute n.